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To cite this version:

Mayssam Mohamad, Jad Dabaghi, Frédéric Grondin, Mazen Saad. Energy estimates and convergence analysis of a two-phase flow in deformable porous media. 2026. ⟨hal-05584989⟩

HAL Id: hal-05584989
https://hal.science/hal-05584989v1

Preprint submitted on 8 Apr 2026

Energy estimates and convergence analysis of a two-phase flow in deformable porous media

Mayssam Mohamad*† Jad Dabaghi‡ Frédéric Grondin† Mazen Saad*

February 2026

Abstract

In this work, we explore two-phase non-isothermal flows in deformable porous media. We consider a Thermo-Hydro-Mechanical (THM) model for a two-phase flow where small deformations and linear thermo-poro-elastic constitutive laws are assumed. These models are widely used in various areas of geomechanics, with applications ranging from underground energy storage to oil and gas reservoir engineering. We present the mathematical formulation of this model which is formulated as a strongly nonlinear system of parabolic partial differential equations governing the conservation of mass, conservation of entropy and momentum balance. Moreover, we derive some energy estimates of the continuous model. The discretization of our system relies on the backward Euler scheme in time and the finite volume two-point flux approximation (TPFA) scheme in space. We show that the energy estimates are well-preserved at the discrete level. These stability results allow us to establish the convergence of the proposed scheme to a weak solution of the nonlinear system. The proof is completed for the degenerate THM model.

Keywords: Thermo-Hydro-Mechanical (THM) model, two-phase flow, energy estimates, finite volume method, convergence analysis.

1 Introduction

The storage of hydrogen, produced via water electrolysis, in a cementitious cavity offers a solution to the overproduction of electricity from wind farms [25, 40]. However, chemical degradation, structural damage, loss of mechanical strength, and an increased leak risk could be caused by hydrogen infiltration into the materials [41, 39, 37, 32]. These challenges highlight the need of constructing and simulating mathematical models for multiphase flows in porous media while taking into account the temperature changes and mechanical behaviors.

In this work, we focus on the Thermo-Hydro-Mechanical (THM) model, following Coussy [18] and Biot [8], for two-phase flow where small deformations, porosity variations and linear thermo-poro-elastic behaviors are assumed. Such models are essential for analyzing fluid flow in deformable porous materials, since they capture the complex interactions between fluid transport, heat transfer, and mechanical deformations within the subsurface. They also provide a framework to evaluate potential risks, such as gas leakage and structural weakening, and to assess different risk mitigation strategies.

Moreover, THM models have been recently the focus for many mathematical and numerical analysis works. For instance, for incompressible single-phase flows, we mention [23] where the Hybrid Finite Volume (HFV) method [22] combined with the finite element method [12, 26] were employed for the numerical discretization. For compressible single phase flows, we refer to the recent work [35] where the finite volume method based on the Two-Point Flux Approximation (TPFA) [28] is employed. In addition, in the context

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[*] Nantes Université, Ecole Centrale Nantes, CNRS, Laboratoire de Mathématiques Jean Leray, LMJL, UMR 6629, F-44000 Nantes, France

[†] Nantes Université, Ecole Centrale Nantes, CNRS, GeM, UMR 6183, F-44000 Nantes, France

[‡] Léonard de Vinci Pôle Universitaire, Research Center, 92 916 Paris La Défense, France

of fractured porous media, we mention [24] where the authors employed a HFV approach to discretize the flow and heat transfer equations, combined with a finite element approach for the mechanical equation. As for two-phase flow in deformable porous media, there exists some contributions for isothermal flows and we mention for instance [10] for a theoretical analysis where the existence of global weak solution was established and [9] for a numerical analysis using the gradient discretization method [21]. Moreover, we refer to [34, 36] for the numerical analysis of poroelastic models using the finite volume method. Next, multiphase flows in rigid porous media where in general the unknowns are the pressure and saturation of the phases (see [17]) have been intensively studied for decades. In Chavent and Jaffré [16], a reduction of these two-phase equations to a system of a single parabolic saturation equation coupled with an elliptic pressure equation is introduced, replacing the two pressure unknowns (one per phase) by only one pressure unknown, called the global pressure. This approach has been widely developped for other theoretical and numerical studies such as [4, 2].

For industrial applications such as the storage of radioactive waste in deep geological layers, we refer to Jaffré and Sboui [33], Ben Gharbia and Jaffré [6], and Ben Gharbia *et al.* [29]. In particular, in the last reference, an a *a posteriori* error estimate for a two phase compositional flow with phase transition is established. Besides, an adaptive procedure following the methodology of [27, 19, 20] is developed in order to distinguish the different error components within the simulation, namely the finite volume discretization error and the semismooth Newton linearization error. We emphasize on the fact that finite volume methods based on the Two-Point Flux Approximation (TPFA) [28] have been widely applied to multiphase flows in rigid porous media, where mechanical deformations are neglected, due to their computational efficiency and conservation properties [38, 30, 1, 11, 7]. While this approach has demonstrated efficiency and accuracy in rigid porous media, its extension to fully coupled THM problems remains challenging, particularly in maintaining energy stability and robustness under large parameter scales. To the best of our knowledge, no theoretical or numerical analysis for the fully coupled THM models with two fluids have been developed before. We will try to fill this gap.

In this paper, we focus on the study of a non-isothermal, immiscible, incompressible two-phase Darcy flow while taking into account the poromechanical coupling. Following [18, 9], we extend the single-phase THM model presented in [35] to the two-phase setting. The resulting mathematical model is a nonlinear degenerate system of partial differential equations involving mass conservation, entropy conservation and momentum conservation with some additional closure equations involving capillary pressure law, linear thermo-poro-elastic constitutive laws and the notion of equivalent pore pressure as defined by Coussy [18]. We first establish the energy estimates for this model. To this end, we introduce a set of assumptions on the physical data, the porosity function and the internal energy of the fluids. A major difficulty arises from the degeneracy of certain dissipative terms, due to the vanishing of the mobilities when the corresponding saturation reaches zero. This issue motivates the use of the global pressure formulation introduced in [16].

For the numerical discretization, we consider the TPFA finite volume scheme for the space discretization together with the implicit Euler scheme for the time discretization. Our first objective is to show that the energy estimates are preserved at the discrete level. The choice of discrete mobilities plays a crucial role in establishing this result. In particular, we adapt an upwind discretization, which enables us to prove a maximum principle for the discrete saturations and ensures that they remain in their physical bounds.

Another main result of this work is the analysis of convergence of the discrete solution to the weak solution of the two-phase THM model. Following the techniques developed in [28, 3, 38], the convergence is weak in general while strong convergence of certain terms is obtained through compactness arguments.

Therefore, this paper is organized as follows. In Section 2, we present the mathematical model of our problem, where an energy-based reformulation of the entropy conservation equation is also presented. Next, in Section 3, we derive the energy estimates for the continuous model and in Section 4, we define the weak solution for our model. Furthermore, in Section 5, we introduce the numerical discretization for our system with the implicit Euler scheme in time, and the cell-centered finite volume scheme in space. Then, we demonstrate that our finite volume scheme is equivalent to a discrete variational formulation. Furthermore, the discrete energy estimates are derived in Section 6. Finally, in Section 7, we prove the convergence of our numerical scheme.

2 Mathematical model

Let $\Omega \subset \mathbb{R}^d$, $d \geq 1$, be an open bounded connected domain representing a porous medium characterized by small strains, displacements, and variations of the rocks. We denote by $t_F > 0$ the final simulation time and we set $Q_{t_F} := (0, t_F) \times \Omega$.

We consider a non-isothermal incompressible two-phase flow in the porous medium Ω. Moreover, we consider a wetting phase, denoted by "w", containing the water component and a non-wetting phase, denoted by "nw", containing the hydrogen component. For a given phase $\alpha \in \{\mathrm{w}, \mathrm{nw}\}$, s_α denotes its saturation, p_α its pressure, c_α its fluid specific entropy, and ρ_α its fluid density. Moreover, T denotes the system temperature, and $\boldsymbol{u}$ the displacement of the skeleton. We extend the Thermo-Hydro-Mechanical (THM) model presented in [35] to the two-phase, non-isothermal, incompressible flow. Linear isotropic thermo-poro-elastic constitutive laws are considered for the skeleton. We assume small variations of temperature around the reference temperature T_{ref} and thermal equilibrium is assumed between the fluids and the skeleton. Here, the primary unknowns of the model are p_α, T, and $\boldsymbol{u}$. Furthermore, the porous medium is characterized by its porosity ϕ, which is a nonlinear function of the primary unknowns, and its absolute permeability tensor denoted by $\mathbb{K} \in \mathbb{R}^{d,d}$. The governing system of partial differential equations consists of mass conservation for each fluid, entropy conservation under the assumption of reversible mechanical deformations, and the momentum balance equation for the skeleton. The model is described in Q_{t_F} by:

$$\partial_t(s_{\mathrm{w}}\rho_{\mathrm{w}}\phi) + \operatorname{div}\left(\rho_{\mathrm{w}}\boldsymbol{V}_{\mathrm{w}}(p_{\mathrm{w}})\right) = r_{\mathrm{w}}, \tag{2.1a}$$

$$\partial_t(s_{\mathrm{nw}}\rho_{\mathrm{nw}}\phi) + \operatorname{div}\left(\rho_{\mathrm{nw}}\boldsymbol{V}_{\mathrm{nw}}(p_{\mathrm{nw}})\right) = r_{\mathrm{nw}}, \tag{2.1b}$$

$$\partial_t\left(S_s + \sum_{\alpha\in\{\mathrm{w},\mathrm{nw}\}} \rho_\alpha s_\alpha c_\alpha \phi\right) + \operatorname{div}\left(\sum_{\alpha\in\{\mathrm{w},\mathrm{nw}\}} \rho_\alpha c_\alpha \boldsymbol{V}_\alpha(p_\alpha) + \frac{1}{T_{\mathrm{ref}}}\boldsymbol{q}(T)\right) = \frac{r_e}{T}, \tag{2.1c}$$

$$m_0\partial_{tt}^2\boldsymbol{u} - \operatorname{div}\boldsymbol{\sigma} = \boldsymbol{f}_u, \tag{2.1d}$$

where S_s is the volumetric skeleton entropy, m_0 is the specific average fluid-rock density, r_{w}, r_{nw} are the source terms related to the fluids, r_e is the external heat rate, and $\boldsymbol{f}_u$ is the body force.

Moreover, we consider homogeneous Neumann boundary conditions for the pressures and temperature variables and homogeneous Dirichlet boundary conditions for the displacement of the skeleton. They are prescribed by:

$$\boldsymbol{V}_{\mathrm{w}} \cdot \boldsymbol{n} = \boldsymbol{V}_{\mathrm{nw}} \cdot \boldsymbol{n} = 0, \quad \boldsymbol{\nabla} T \cdot \boldsymbol{n} = 0, \quad \boldsymbol{u} = 0 \qquad \text{in } (0, t_F) \times \partial\Omega,$$

with $\boldsymbol{n}$ the outward unit normal vector to Ω. At $t = 0$, the initial data are prescribed by:

$$p_{\mathrm{w}}(.,0) = p_{\mathrm{w}}^0, \quad p_{\mathrm{nw}}(.,0) = p_{\mathrm{nw}}^0, \quad T(.,0) = T^0, \quad \boldsymbol{u}(.,0) = \boldsymbol{u}^0 \qquad \text{in } \Omega.$$

The Darcy velocity $\boldsymbol{V}_\alpha$ for each phase $\alpha \in \{\mathrm{w}, \mathrm{nw}\}$ is defined by

$$\boldsymbol{V}_\alpha := -\frac{K_{r\alpha}(s_\alpha)}{\mu_\alpha}\mathbb{K}(\boldsymbol{\nabla} p_\alpha - \rho_\alpha \mathrm{g} \boldsymbol{\nabla} z), \tag{2.2}$$

where $K_{r\alpha}$ is the relative permeability of the phase α, μ_α the dynamic viscosity of the fluid α, and g is the gravity acceleration constant. Moreover, we refer to the ratio $K_{r\alpha}/\mu_\alpha$ as the mobility of the phase $\alpha \in \{\mathrm{w}, \mathrm{nw}\}$, denoted by M_α depending only on the saturation s_α.

We assume that the medium is totally filled by the two fluids, which means that

$$s_{\mathrm{w}} + s_{\mathrm{nw}} = 1. \tag{2.3}$$

The relation between the non-wetting phase nw and the wetting phase w, is determined by the capillary pressure, which is defined by

$$p_{\mathrm{c}}(s_{\mathrm{w}}) := p_{\mathrm{nw}} - p_{\mathrm{w}}. \tag{2.4}$$

Classical examples of the capillary pressure include the Van Genuchten model [42] and the Brooks–Corey model [14]. Moreover, the conductive heat flux $\boldsymbol{q}$ is defined by Fourier's law as

$$\boldsymbol{q} := -\lambda \boldsymbol{\nabla} T, \tag{2.5}$$

where λ is the fluid rock average thermal conductivity.

The skeleton's stress is modeled using a linear isotropic thermo-poro-elastic constitutive relation. The symmetric total-stress tensor $\boldsymbol{\sigma}$ is defined from the effective stress tensor $\boldsymbol{\sigma}^e$ by

$$\begin{aligned} \boldsymbol{\sigma}(\boldsymbol{u}, p_{\mathrm{w}}, p_{\mathrm{nw}}, T) &:= \boldsymbol{\sigma}^e(\boldsymbol{u}) - b\pi \mathbb{I}_d - 3\alpha_s K_s (T - T_{\mathrm{ref}})\mathbb{I}_d, \\ \boldsymbol{\sigma}^e(\boldsymbol{u}) &:= \frac{\mathcal{E}}{1+\nu}\left(\boldsymbol{\epsilon}(\boldsymbol{u}) + \frac{\nu}{1-2\nu}\operatorname{div}\boldsymbol{u}\mathbb{I}_d\right), \end{aligned} \tag{2.6}$$

where $\mathcal{E}$ is the effective Young's modulus, ν is the Poisson coefficient, b is the Biot coefficient, K_s is the bulk modulus, and $3\alpha_s$ is the volumetric skeleton thermal dilatation coefficient. Moreover, π denotes the equivalent pore pressure (we refer to [18] for more details) which is defined by:

$$\pi := p^\star - U, \tag{2.7}$$

where $p^\star$ is the averaged fluid pressure defined by:

$$p^\star := s_{\mathrm{w}} p_{\mathrm{w}} + s_{\mathrm{nw}} p_{\mathrm{nw}},$$

and U is the interfacial energy, which is a nonnegative function of the saturation s_{w} and it is defined by:

$$U(s_{\mathrm{w}}) := \int_{s_{\mathrm{w}}}^{1} p_{\mathrm{c}}(z)\, dz. \tag{2.8}$$

In addition, $\boldsymbol{\epsilon}(\boldsymbol{u}) \in \mathbb{R}^{d,d}$ denotes the strain tensor defined by

$$\boldsymbol{\epsilon}(\boldsymbol{u}) := \frac{1}{2}(\boldsymbol{\nabla}\boldsymbol{u} + (\boldsymbol{\nabla}\boldsymbol{u})^\top),$$

and $\mathbb{I}_d \in \mathbb{R}^{d,d}$ is the identity matrix.

The porosity ϕ and the volumetric skeleton entropy S_s can be modeled as

$$\begin{aligned} \partial_t \phi &= b\partial_t(\operatorname{div}\boldsymbol{u}) - 3\alpha_\phi \partial_t T + \frac{1}{N}\partial_t \pi, \\ \partial_t S_s &= 3\alpha_s K_s \partial_t(\operatorname{div}\boldsymbol{u}) - 3\alpha_\phi \partial_t \pi + \frac{C_s}{T_{\mathrm{ref}}}\partial_t T, \end{aligned} \tag{2.9}$$

where N is the Biot modulus, $3\alpha_\phi$ is the volumetric thermal dilatation coefficient related to the porosity, and C_s is the skeleton volumetric heat capacity.

Furthermore, the energy balance in the system is governed by the laws of thermodynamics, which relate entropy, energy and density variations for each phase $\alpha \in \{\mathrm{w}, \mathrm{nw}\}$. Specifically, since the fluids are incompressible, we have the following entropy-energy relation:

$$\partial_t e_\alpha = T\partial_t c_\alpha, \quad \boldsymbol{\nabla} e_\alpha = T\boldsymbol{\nabla} c_\alpha, \tag{2.10}$$

where for a given phase $\alpha \in \{\mathrm{w}, \mathrm{nw}\}$, c_α denotes its fluid-specific entropy and e_α its internal energy. Consequently, the entropy conservation equation (2.1c) can be rewritten as:

$$\partial_t S_s + \sum_{\alpha \in \{\mathrm{w},\mathrm{nw}\}} \left[\frac{\phi \rho_\alpha s_\alpha}{T}\partial_t e_\alpha + \frac{\rho_\alpha \boldsymbol{V}_\alpha(p_\alpha)}{T}\cdot \boldsymbol{\nabla} e_\alpha\right] + \frac{1}{T_{\mathrm{ref}}}\operatorname{div}\boldsymbol{q}(T) = \frac{r_e}{T} - c_{\mathrm{w}} r_{\mathrm{w}} - c_{\mathrm{nw}} r_{\mathrm{nw}}. \tag{2.11}$$

In fact, substituting the fluid entropy-energy relation (2.10) into the entropy conservation equation (2.1c), we obtain:

$$\begin{aligned} \partial_t S_s &+ \sum_{\alpha \in \{\mathrm{w},\mathrm{nw}\}} \left[\frac{\phi \rho_\alpha s_\alpha}{T}\partial_t e_\alpha + c_\alpha \rho_\alpha \partial_t(\phi s_\alpha)\right] \\ &+ \sum_{\alpha \in \{\mathrm{w},\mathrm{nw}\}} \left[c_\alpha \rho_\alpha \operatorname{div}(\boldsymbol{V}_\alpha(p_\alpha)) + \frac{\rho_\alpha \boldsymbol{V}_\alpha(p_\alpha)}{T}\cdot \boldsymbol{\nabla} e_\alpha\right] + \frac{1}{T_{\mathrm{ref}}}\operatorname{div}\boldsymbol{q}(T) = \frac{r_e}{T}. \end{aligned}$$

Moreover, employing the mass conservation equations (2.1a)-(2.1b), we recover the reformulated entropy conservation equation (2.11).

2.1 Global Pressure

In this section, we recall the notion of the global pressure introduced by [16] in order to handle the degeneracy of the gradient terms in the system of equations (2.1) due to the vanishing of the mobilities when the corresponding saturation is zero and to be able to obtain the energy estimates.

We recall the definition of the global pressure denoted by p as described in [16]

$$M(s_{\mathrm{w}})\boldsymbol{\nabla} p = M_{\mathrm{w}}(s_{\mathrm{w}})\boldsymbol{\nabla} p_{\mathrm{w}} + M_{\mathrm{nw}}(s_{\mathrm{nw}})\boldsymbol{\nabla} p_{\mathrm{nw}}.$$

Where $M(s_{\mathrm{w}})$ is the total mobility and it is defined by

$$M(s_{\mathrm{w}}) = M_{\mathrm{w}}(s_{\mathrm{w}}) + M_{\mathrm{nw}}(s_{\mathrm{nw}}).$$

Moreover, the global pressure p can be written as

$$p = p_{\mathrm{w}} + \hat{p}_{\mathrm{w}}(s_{\mathrm{w}}) = p_{\mathrm{nw}} + \hat{p}_{\mathrm{nw}}(s_{\mathrm{w}}),$$

where the corrective pressures $\hat{p}_{\mathrm{w}}$, $\hat{p}_{\mathrm{nw}}$ are defined as

$$\hat{p}_{\mathrm{w}}(s_{\mathrm{w}}) = -\int_0^{s_{\mathrm{w}}} \frac{M_{\mathrm{nw}}(1-z)}{M(z)} p_{\mathrm{c}}'(z)\,\mathrm{d}z \quad \text{and} \quad \hat{p}_{\mathrm{nw}}(s_{\mathrm{w}}) = \int_0^{s_{\mathrm{w}}} \frac{M_{\mathrm{w}}(z)}{M(z)} p_{\mathrm{c}}'(z)\,\mathrm{d}z.$$

Furthermore, we define the following capillary term $\mathcal{B}$, known as the Kirchhoff transform, by

$$\mathcal{B}(s_{\mathrm{w}}) := \int_0^{s_{\mathrm{w}}} -\frac{M_{\mathrm{w}}(z)M_{\mathrm{nw}}(1-z)}{M(z)} p_{\mathrm{c}}'(z)\,\mathrm{d}z. \tag{2.12}$$

From these definitions, we have the following identities

$$M_{\mathrm{w}}(s_{\mathrm{w}})\boldsymbol{\nabla} p_{\mathrm{w}} = M_{\mathrm{w}}(s_{\mathrm{w}})\boldsymbol{\nabla} p - \boldsymbol{\nabla}\mathcal{B}(s_{\mathrm{w}}), \quad M_{\mathrm{nw}}(s_{\mathrm{nw}})\boldsymbol{\nabla} p_{\mathrm{nw}} = M_{\mathrm{nw}}(s_{\mathrm{nw}})\boldsymbol{\nabla} p + \boldsymbol{\nabla}\mathcal{B}(s_{\mathrm{w}}).$$

Consequently, we obtain the following equality

$$M_{\mathrm{w}}(s_{\mathrm{w}})|\boldsymbol{\nabla} p_{\mathrm{w}}|^2 + M_{\mathrm{nw}}(s_{\mathrm{nw}})|\boldsymbol{\nabla} p_{\mathrm{nw}}|^2 = M(s_{\mathrm{w}})|\boldsymbol{\nabla} p|^2 + |\boldsymbol{\nabla}\mathcal{B}(s_{\mathrm{w}})|^2. \tag{2.13}$$

2.2 Assumptions

Furthermore, we present some key assumptions on the physical data in order to obtain energy estimates on the system.

(A1) The porosity ϕ is always positive, i.e. there exists $\phi_* \in \mathbb{R}_+^*$ such that $\phi(\boldsymbol{u}, p_{\mathrm{w}}, p_{\mathrm{nw}}, T) \geq \phi_* > 0$.

(A2) The energy functional is non-negative, meaning that for each phase $\alpha \in \{\mathrm{w}, \mathrm{nw}\}$, $e_\alpha(T) \geq 0\ \forall T \in \mathbb{R}_+$ and $e_\alpha - Tc_\alpha$ is sub-quadratic in the sense that

$$\lim_{|T|\to+\infty} \frac{e_\alpha - Tc_\alpha}{|T|^2} = 0.$$

(A3) The source terms r_{w}, r_{nw}, r_e and $\boldsymbol{f}_u$ are bounded. Moreover, r_{w} and r_{nw} are non-negative.

(A4) The thermo-poro-elastic coefficients satisfy the following conditions: $C_s > 0$, $\dfrac{1}{N} > 0$, $\alpha_\phi \geq 0$, $\mathcal{E} > 0$ and $\nu \in]0, \frac{1}{2}[$.

(A5) The specific average fluid-rock density m_0 is positive, i.e, $m_0 > 0$.

(A6) The mobility M_α of the phase $\alpha \in \{\mathrm{w}, \mathrm{nw}\}$ is a continuous, non-decreasing function from $[0, 1]$ to $\mathbb{R}$ such that $M_\alpha(0) = 0$. Moreover, we take into account the extension by continuity outside $[0, 1]$: $M_\alpha(s_\alpha) = 0$ for every $s_\alpha \in]-\infty, 0]$. In addition, the total mobility is bounded far away from 0, i.e, there exists a positive constant $M_0 > 0$ such that for all $s_{\mathrm{w}} \in [0, 1]$,

$$M(s_{\mathrm{w}}) = M_{\mathrm{w}}(s_{\mathrm{w}}) + M_{\mathrm{nw}}(1 - s_{\mathrm{w}}) \geq M_0.$$

(A7) The capillary function $p_{\mathrm{c}}(s_{\mathrm{w}}) \in C^1([0, 1], \mathbb{R}^+)$ is decreasing and we assume that $p_{\mathrm{c}}(1) = 0$.

(A8) The permeability tensor $\mathbb{K}$ is symmetric and uniformly elliptic on Ω.

(A9) The initial pressures, temperature, porosity and skeleton entropy are such that $p_\alpha^0 \in L^2(\Omega)$, $\alpha \in \{\mathrm{w}, \mathrm{nw}\}$, $T^0 \in L^2(\Omega)$, $\phi^0 \in]0, 1[$ and $S_s^0 \in L^\infty(\Omega)$.

3 Energy estimates for the continuous model

In this section, we derive some energy estimates for the main variables, namely the pressures, temperature, displacement, and their respective gradients. To this end, we define the function $\mathcal{N} : [0,1] \to \mathbb{R}$ by

$$\mathcal{N}(s_{\mathrm{w}}) := \int_{s_{\mathrm{w}}}^{1} z\, p_{\mathrm{c}}'(z)\,\mathrm{d}z. \tag{3.1}$$

Following the notations in [31], we recall that the trace operator $tr : \mathbb{R}^{m,m} \to \mathbb{R}$ is defined by $tr(\mathbb{A}) := \sum_{i=1}^{m} a_{ii}$, with $\mathbb{A} := (a_{ij})_{1\le i,j\le m}$ is a square matrix. The contracted product of two tensors $\mathbb{A}$ and $\mathbb{B}$, both of order n, is denoted by $\mathbb{A}\cdot\mathbb{B}$ and is defined as the tensor with entries $(\mathbb{A}\cdot\mathbb{B})_{ij} := \sum_{k=1}^{n} a_{ik}b_{kj}$. The double contracted product of $\mathbb{A}$ and $\mathbb{B}$, denoted by $\mathbb{A} : \mathbb{B}$, is a scalar given by $\mathbb{A} : \mathbb{B} := tr\,(\mathbb{A}\cdot\mathbb{B}) = \sum_{i,j=1}^{n} a_{ij}b_{ij}$. For any vectors $\boldsymbol{v}$ and $\boldsymbol{w} \in \mathbb{R}^m$, we define their tensor product $\boldsymbol{v}\otimes\boldsymbol{w}$ as the $m\times m$ matrix whose (i,j)-th entry is given by $(\boldsymbol{v}\otimes\boldsymbol{w})_{ij} := v_i w_j$. Finally, the norm of a tensor $\mathbb{A}$ is denoted by $|\mathbb{A}|$ and is defined by $|\mathbb{A}| := (\mathbb{A} : \mathbb{A})^{1/2}$.

Furthermore, in the sequel we neglect the gravitational term g for the sake of simplicity. We state the following lemma which provides a key identity that will be used to derive the energy estimates.

Lemma 1. *Let $(\boldsymbol{u}, p_{\mathrm{w}}, p_{\mathrm{nw}}, T)$ be the solution of the system* (2.1)*. We assume that the system of equations* (2.1) *is subject to Neumann boundary conditions for the fluxes and Dirichlet boundary conditions for the displacement of the skeleton. The following equality holds*

$$\begin{aligned}
&\int_{Q_{t_F}} \partial_t E\,\mathrm{d}\boldsymbol{x}\mathrm{d}t + \int_{Q_{t_F}} \frac{m_0}{2}\partial_t\left(|\partial_t\boldsymbol{u}|^2\right)\mathrm{d}\boldsymbol{x}\mathrm{d}t + \int_{Q_{t_F}} \frac{\lambda}{T_{\mathrm{ref}}}|\boldsymbol{\nabla}T|^2\,\mathrm{d}\boldsymbol{x}\mathrm{d}t\\
&+\int_{Q_{t_F}} \left(M_{\mathrm{w}}(s_{\mathrm{w}})\mathbb{K}\boldsymbol{\nabla}p_{\mathrm{w}}\cdot\boldsymbol{\nabla}p_{\mathrm{w}} + M_{\mathrm{nw}}(s_{\mathrm{nw}})\mathbb{K}\boldsymbol{\nabla}p_{\mathrm{nw}}\cdot\boldsymbol{\nabla}p_{\mathrm{nw}}\right)\mathrm{d}\boldsymbol{x}\mathrm{d}t\\
&=\int_{Q_{t_F}} \left(\sum_{\alpha\in\{\mathrm{w},\mathrm{nw}\}} r_\alpha\left(\frac{p_\alpha}{\rho_\alpha} + e_\alpha - Tc_\alpha\right) + r_e + \boldsymbol{f}_u\cdot\partial_t\boldsymbol{u}\right)\mathrm{d}\boldsymbol{x}\mathrm{d}t,
\end{aligned}\tag{3.2}$$

where E is the total energy of the system defined by

$$E := \phi U + \frac{1}{2}\begin{bmatrix}\pi & T\end{bmatrix}\mathbb{M}\begin{bmatrix}\pi\\ T\end{bmatrix} + \frac{\mathcal{E}}{2(1+\nu)}\left[|\epsilon(\boldsymbol{u})|^2 + \frac{\nu}{1-2\nu}(\operatorname{div}\boldsymbol{u})^2\right] + \sum_{\alpha\in\{\mathrm{w},\mathrm{nw}\}} \phi s_\alpha\rho_\alpha e_\alpha, \tag{3.3}$$

and $\mathbb{M}\in\mathbb{R}^{2,2}$ denotes a real, symmetric, positive definite matrix defined as

$$\mathbb{M} := \begin{bmatrix}\frac{1}{N} & -3\alpha_\phi\\ -3\alpha_\phi & \frac{C_0}{T_{\mathrm{ref}}}\end{bmatrix}. \tag{3.4}$$

Proof of Lemma 1. We multiply equations (2.1a), (2.1b), (2.11) and (2.1d) by $\left(\frac{p_{\mathrm{w}}}{\rho_{\mathrm{w}}} + e_{\mathrm{w}}\right)$, $\left(\frac{p_{\mathrm{nw}}}{\rho_{\mathrm{nw}}} + e_{\mathrm{nw}}\right)$, T and $\partial_t\boldsymbol{u}$ respectively. Then, we integrate over space and time, apply the divergence theorem and sum the resulting equations to obtain

$$\begin{aligned}
&\int_{Q_{t_F}} \left(\partial_t(s_{\mathrm{w}}\phi)p_{\mathrm{w}} + \partial_t(s_{\mathrm{nw}}\phi)p_{\mathrm{nw}}\right)\mathrm{d}\boldsymbol{x}\mathrm{d}t + \int_{Q_{t_F}} \partial_t(\rho_{\mathrm{w}}s_{\mathrm{w}}\phi e_{\mathrm{w}} + \rho_{\mathrm{nw}}s_{\mathrm{nw}}\phi e_{\mathrm{nw}})\,\mathrm{d}\boldsymbol{x}\mathrm{d}t\\
&+\int_{Q_{t_F}} T\partial_t S_s\,\mathrm{d}\boldsymbol{x}\mathrm{d}t - \int_{Q_{t_F}} \boldsymbol{V}_{\mathrm{w}}(p_{\mathrm{w}})\cdot\boldsymbol{\nabla}p_{\mathrm{w}}\,\mathrm{d}\boldsymbol{x}\mathrm{d}t - \int_{Q_{t_F}} \boldsymbol{V}_{\mathrm{nw}}(p_{\mathrm{nw}})\cdot\boldsymbol{\nabla}p_{\mathrm{nw}}\,\mathrm{d}\boldsymbol{x}\mathrm{d}t\\
&-\int_{Q_{t_F}} \frac{1}{T_{\mathrm{ref}}}\boldsymbol{q}(T)\cdot\boldsymbol{\nabla}T\,\mathrm{d}\boldsymbol{x}\mathrm{d}t + \int_{Q_{t_F}} m_0\partial_{tt}^2\boldsymbol{u}\cdot\partial_t\boldsymbol{u}\,\mathrm{d}\boldsymbol{x}\mathrm{d}t + \int_{Q_{t_F}} \sigma^e(\boldsymbol{u}) : \epsilon(\partial_t\boldsymbol{u})\,\mathrm{d}\boldsymbol{x}\mathrm{d}t\\
&-\int_{Q_{t_F}} (b\pi + 3\alpha_s K_s T)\operatorname{div}(\partial_t\boldsymbol{u})\,\mathrm{d}\boldsymbol{x}\mathrm{d}t\\
&=\int_{Q_{t_F}} \left(r_e + r_{\mathrm{w}}\left(\frac{p_{\mathrm{w}}}{\rho_{\mathrm{w}}} - Tc_{\mathrm{w}} + e_{\mathrm{w}}\right) + r_{\mathrm{nw}}\left(\frac{p_{\mathrm{nw}}}{\rho_{\mathrm{nw}}} - Tc_{\mathrm{nw}} + e_{\mathrm{nw}}\right) + \boldsymbol{f}_u\cdot\partial_t\boldsymbol{u}\right)\mathrm{d}\boldsymbol{x}\mathrm{d}t.
\end{aligned}\tag{3.5}$$

From the first term of (3.5), we obtain

$$\begin{aligned} \partial_t(s_\mathrm{w}\phi)p_\mathrm{w} + \partial_t(s_\mathrm{nw}\phi)p_\mathrm{nw} &= p^\star\partial_t\phi + \phi p_\mathrm{w}\partial_t s_\mathrm{w} + \phi p_\mathrm{nw}\partial_t s_\mathrm{nw} \\ &= p^\star\partial_t\phi - \phi p_\mathrm{c}(s_\mathrm{w})\partial_t s_\mathrm{w} \\ &= p^\star\partial_t\phi - \phi\partial_t(p_\mathrm{c}(s_\mathrm{w})s_\mathrm{w}) + \phi s_\mathrm{w} p_\mathrm{c}'(s_\mathrm{w})\partial_t s_\mathrm{w}. \end{aligned} \tag{3.6}$$

Moreover, using the definition of the function $\mathcal{N}$ given by (3.1) together with the definition of the interfacial energy U given by (2.8) and Assumption (**A7**), we obtain

$$\begin{aligned} &-\phi\partial_t(p_\mathrm{c}(s_\mathrm{w})s_\mathrm{w}) + \phi s_\mathrm{w} p_\mathrm{c}'(s_\mathrm{w})\partial_t s_\mathrm{w} = -\phi\partial_t(p_\mathrm{c}(s_\mathrm{w})s_\mathrm{w}) - \phi\partial_t(\mathcal{N}(s_\mathrm{w})) \\ &= \phi\partial_t\left(\int_{s_\mathrm{w}}^1 p_\mathrm{c}(z)\,\mathrm{d}z\right) = \phi\partial_t(U(s_\mathrm{w})). \end{aligned} \tag{3.7}$$

Consequently, combining (3.6) and (3.7) and using (2.7), the first term of (3.5) reads

$$\begin{aligned} &\int_{Q_{t_F}} \left(\partial_t(s_\mathrm{w}\phi)p_\mathrm{w} + \partial_t(s_\mathrm{nw}\phi)p_\mathrm{nw}\right)\mathrm{d}\boldsymbol{x}\mathrm{d}t = \int_{Q_{t_F}} \left(p^\star\partial_t\phi + \phi\partial_t(U(s_\mathrm{w}))\right)\mathrm{d}\boldsymbol{x}\mathrm{d}t \\ &= \int_{Q_{t_F}} \left(p^\star\partial_t\phi + \partial_t(U(s_\mathrm{w})\phi) - U(s_\mathrm{w})\partial_t\phi\right)\mathrm{d}\boldsymbol{x}\mathrm{d}t = \int_{Q_{t_F}} \left(\pi\partial_t\phi + \partial_t(U(s_\mathrm{w})\phi)\right)\mathrm{d}\boldsymbol{x}\mathrm{d}t. \end{aligned} \tag{3.8}$$

Furthermore, employing Darcy's velocity from definition (2.2), we obtain

$$\boldsymbol{V}_\mathrm{w}\cdot\boldsymbol{\nabla}p_\mathrm{w} + \boldsymbol{V}_\mathrm{nw}\cdot\boldsymbol{\nabla}p_\mathrm{nw} = -\left(M_\mathrm{w}(s_\mathrm{w})\mathbb{K}\boldsymbol{\nabla}p_\mathrm{w}\cdot\boldsymbol{\nabla}p_\mathrm{w} + M_\mathrm{nw}(s_\mathrm{nw})\mathbb{K}\boldsymbol{\nabla}p_\mathrm{nw}\cdot\boldsymbol{\nabla}p_\mathrm{nw}\right). \tag{3.9}$$

In addition, employing the definition of the heat flux (2.5), the sixth term of (3.5) reads

$$\boldsymbol{q}(T)\cdot\boldsymbol{\nabla}T = -\lambda|\boldsymbol{\nabla}T|^2. \tag{3.10}$$

Using the identity $v\partial_t v = \frac{1}{2}\partial_t(v^2)$, the seventh term of (3.5) reads

$$m_0\partial_{tt}^2\boldsymbol{u}\cdot\partial_t\boldsymbol{u} = \frac{m_0}{2}\partial_t(\partial_t\boldsymbol{u}\cdot\partial_t\boldsymbol{u}) = \frac{m_0}{2}\partial_t\left(|\partial_t\boldsymbol{u}|^2\right). \tag{3.11}$$

From the definition of the effective stress tensor (2.6), we have

$$\sigma^e(\boldsymbol{u}):\epsilon(\partial_t\boldsymbol{u}) = \frac{\mathcal{E}}{2(1+\nu)}\partial_t\left(|\epsilon(\boldsymbol{u})|^2\right) + \frac{\mathcal{E}\nu}{2(1+\nu)(1-2\nu)}\partial_t(\operatorname{div}\boldsymbol{u})^2. \tag{3.12}$$

Moreover, employing the definitions of the porosity and skeleton entropy (2.9), we obtain

$$\begin{aligned} &\pi\partial_t(\phi) + T\partial_t(S_s) - (b\pi + 3\alpha_s K_s T)\operatorname{div}(\partial_t\boldsymbol{u}) \\ &= -3\alpha_\phi\pi\partial_t T - 3\alpha_\phi T\partial_t\pi + \frac{1}{2N}\partial_t\pi^2 + \frac{C_s}{2T_\mathrm{ref}}\partial_t T^2 \\ &= \partial_t\left[\frac{1}{2}\begin{bmatrix}\pi & T\end{bmatrix}\begin{bmatrix}\frac{1}{N} & -3\alpha_\phi \\ -3\alpha_\phi & \frac{C_s}{T_\mathrm{ref}}\end{bmatrix}\begin{bmatrix}\pi \\ T\end{bmatrix}\right]. \end{aligned} \tag{3.13}$$

Finally, combining (3.8), (3.9), (3.10), (3.11), (3.12) and (3.13), equation (3.5) reads

$$\begin{aligned} &\int_{Q_{t_F}} \partial_t(\rho_\mathrm{w}s_\mathrm{w}\phi e_\mathrm{w} + \rho_\mathrm{nw}s_\mathrm{nw}\phi e_\mathrm{nw} + U\phi)\,\mathrm{d}\boldsymbol{x}\mathrm{d}t + \int_{Q_{t_F}} \frac{m_0}{2}\partial_t\left(|\partial_t\boldsymbol{u}|^2\right)\mathrm{d}\boldsymbol{x}\mathrm{d}t \\ &+ \int_{Q_{t_F}} \left(\frac{\mathcal{E}}{2(1+\nu)}\partial_t\left(|\epsilon(\boldsymbol{u})|^2\right) + \frac{\mathcal{E}\nu}{2(1+\nu)(1-2\nu)}\partial_t(\operatorname{div}\boldsymbol{u})^2\right)\mathrm{d}\boldsymbol{x}\mathrm{d}t \\ &+ \int_{Q_{t_F}} \partial_t\left[\frac{1}{2}\begin{bmatrix}\pi & T\end{bmatrix}\begin{bmatrix}\frac{1}{N} & -3\alpha_\phi \\ -3\alpha_\phi & \frac{C_s}{T_\mathrm{ref}}\end{bmatrix}\begin{bmatrix}\pi \\ T\end{bmatrix}\right]\mathrm{d}\boldsymbol{x}\mathrm{d}t + \int_{Q_{t_F}} \frac{\lambda}{T_\mathrm{ref}}|\boldsymbol{\nabla}T|^2\,\mathrm{d}\boldsymbol{x}\mathrm{d}t \\ &+ \int_{Q_{t_F}} \left(M_\mathrm{w}(s_\mathrm{w})\mathbb{K}\boldsymbol{\nabla}p_\mathrm{w}\cdot\boldsymbol{\nabla}p_\mathrm{w} + M_\mathrm{nw}(s_\mathrm{nw})\mathbb{K}\boldsymbol{\nabla}p_\mathrm{nw}\cdot\boldsymbol{\nabla}p_\mathrm{nw}\right)\mathrm{d}\boldsymbol{x}\mathrm{d}t \\ &= \int_{Q_{t_F}} \left(r_e + r_\mathrm{w}\left(\frac{p_\mathrm{w}}{\rho_\mathrm{w}} - Tc_\mathrm{w} + e_\mathrm{w}\right) + r_\mathrm{nw}\left(\frac{p_\mathrm{nw}}{\rho_\mathrm{nw}} - Tc_\mathrm{nw} + e_\mathrm{nw}\right) + \boldsymbol{f}_u\cdot\partial_t\boldsymbol{u}\right)\mathrm{d}\boldsymbol{x}\mathrm{d}t. \end{aligned}$$

Setting,

$$E := \phi U + \frac{1}{2}\begin{bmatrix}\pi & T\end{bmatrix}\mathbb{M}\begin{bmatrix}\pi \\ T\end{bmatrix} + \frac{\mathcal{E}}{2(1+\nu)}\left[|\epsilon(\boldsymbol{u})|^2 + \frac{\nu}{1-2\nu}(\operatorname{div}\boldsymbol{u})^2\right] + \sum_{\alpha\in\{\mathrm{w},\mathrm{nw}\}} \phi s_\alpha \rho_\alpha e_\alpha,$$

with

$$\mathbb{M} := \begin{bmatrix}\frac{1}{N} & -3\alpha_\phi \\ -3\alpha_\phi & \frac{C_s}{T_{\mathrm{ref}}}\end{bmatrix},$$

we recover equation (3.2) which concludes the proof. □

Now, the following result states the energy estimates.

Proposition 1. *Let* $(\boldsymbol{u}, p_{\mathrm{w}}, p_{\mathrm{nw}}, T)$ *be the solution of system* (2.1), $\alpha \in \{\mathrm{w},\mathrm{nw}\}$ *and* e_α *be the fluid-specific energy of the phase* α. *Assume that the assumptions in Section 2.2 are fulfilled. We have the following energy estimate:*

$$\begin{aligned}&\|\partial_t\boldsymbol{u}\|_{L^\infty(0,t_F;(L^2(\Omega))^d)} + \|p_\alpha\|_{L^\infty(0,t_F;L^2(\Omega))} + \|T\|_{L^\infty(0,t_F;L^2(\Omega))} + \|e_\alpha\|_{L^\infty(0,t_F;L^1(\Omega))} \\ &+ \|\boldsymbol{u}\|_{L^\infty(0,t_F;(H_0^1(\Omega))^d)} + \|\boldsymbol{\nabla} p\|_{L^2(0,t_F;(L^2(\Omega))^d)} + \|\boldsymbol{\nabla} T\|_{L^2(0,t_F;(L^2(\Omega))^d)} \\ &+ \|U\|_{L^\infty(0,t_F;L^1(\Omega))} + \|\pi\|_{L^\infty(0,t_F;L^2(\Omega))} \le C.\end{aligned}$$

Proof of Proposition 1. Let $y : (0,t_F) \to \mathbb{R}$ be the function defined by

$$y(t) := \int_\Omega \left(E(t,\boldsymbol{x}) + \frac{m_0}{2}|\partial_t\boldsymbol{u}(t,\boldsymbol{x})|^2\right)\,\mathrm{d}\boldsymbol{x}, \tag{3.14}$$

where E is the energy of the system defined by (3.3). We aim to show that there exist two functions $k_1 : (0,t_F) \to \mathbb{R}$ and $k_2 : (0,t_F) \to \mathbb{R}$ such that

$$\int_0^{t_F} y'(t)\,\mathrm{d}t \le \int_0^{t_F} (k_1(t)y(t) + k_2(t))\,\mathrm{d}t,$$

in order to apply Gronwall's inequality [5, Theorem 1.1] and thereby derive the desired result.

Using the Rayleigh quotient since $\mathbb{M}$ is a real symmetric matrix, we obtain

$$\lambda_{\min}|(\pi,T)|^2 \le \begin{bmatrix}\pi & T\end{bmatrix}\mathbb{M}\begin{bmatrix}\pi \\ T\end{bmatrix} \le \lambda_{\max}|(\pi,T)|^2, \tag{3.15}$$

where $\lambda_{\min}$ and $\lambda_{\max}$ denote, respectively, the minimum and the maximum eigenvalues of $\mathbb{M}$. Furthermore, using Assumptions **(A1)** and **(A4)** together with inequality (3.15) and the fact that $U \ge 0$, the definition of E given by (3.3) yields

$$\begin{aligned} E &\ge \frac{1}{2}\lambda_{\min}|(\pi,T)|^2 + \rho_{\mathrm{w}}\phi_* s_{\mathrm{w}} e_{\mathrm{w}}(T) + \rho_{\mathrm{nw}}\phi_* s_{\mathrm{nw}} e_{\mathrm{nw}}(T) \\ &\ge C_0(|\pi|^2 + |T|^2 + s_{\mathrm{w}} e_{\mathrm{w}}(T) + s_{\mathrm{nw}} e_{\mathrm{nw}}(T)),\end{aligned} \tag{3.16}$$

where $C_0 := \min\left(\frac{1}{2}\lambda_{min}, \rho_{\mathrm{w}}\phi_*, \rho_{\mathrm{nw}}\phi_*\right)$ is a real positive constant. Furthermore, from Lemma 1 and the definition of $y(t)$ given by (3.14), we have

$$\begin{aligned}\int_0^{t_F} y'(t)\,\mathrm{d}t &\le \int_{Q_{t_F}} \partial_t E\,\mathrm{d}\boldsymbol{x}\mathrm{d}t + \int_{Q_{t_F}} \frac{m_0}{2}\partial_t(|\partial_t\boldsymbol{u}|^2)\,\mathrm{d}\boldsymbol{x}\mathrm{d}t + \int_{Q_{t_F}} \frac{\lambda}{T_{\mathrm{ref}}}|\boldsymbol{\nabla} T|^2\,\mathrm{d}\boldsymbol{x}\mathrm{d}t \\ &+ \int_{Q_{t_F}} (M_{\mathrm{w}}(s_{\mathrm{w}})\mathbb{K}\boldsymbol{\nabla} p_{\mathrm{w}} \cdot \boldsymbol{\nabla} p_{\mathrm{w}} + M_{\mathrm{nw}}(s_{\mathrm{nw}})\mathbb{K}\boldsymbol{\nabla} p_{\mathrm{nw}} \cdot \boldsymbol{\nabla} p_{\mathrm{nw}})\,\mathrm{d}\boldsymbol{x}\mathrm{d}t \\ &= \int_{Q_{t_F}} \left(\sum_{\alpha\in\{\mathrm{w},\mathrm{nw}\}} r_\alpha\left(\frac{p_\alpha}{\rho_\alpha} - Tc_\alpha + e_\alpha\right) + r_e + \boldsymbol{f}_u \cdot \partial_t\boldsymbol{u}\right)\mathrm{d}\boldsymbol{x}\mathrm{d}t.\end{aligned} \tag{3.17}$$

By Assumptions **(A2)** and **(A3)**, there exist positive constants $(C_{1,\alpha}, C_2, C_3, C_{4,\alpha}) \in \mathbb{R}_+^4$, $\alpha \in \{\mathrm{w}, \mathrm{nw}\}$ such that

$$|r_\alpha| \le C_{1,\alpha}, \quad |r_e| \le C_2, \quad |\boldsymbol{f}_u| \le C_3, \quad e_\alpha - Tc_\alpha \le C_{4,\alpha}|T|^2. \tag{3.18}$$

Employing the bounds (3.18) and Young's inequality, we obtain the following bound on the right-hand side of (3.17):

$$\begin{aligned}
&\int_{Q_{t_F}} \left(r_{\mathrm{w}} \left(\frac{p_{\mathrm{w}}}{\rho_{\mathrm{w}}} + e_{\mathrm{w}} - c_{\mathrm{w}} T \right) + r_{\mathrm{nw}} \left(\frac{p_{\mathrm{nw}}}{\rho_{\mathrm{nw}}} + e_{\mathrm{nw}} - c_{\mathrm{nw}} T \right) + r_e + \boldsymbol{f}_u \cdot \partial_t \boldsymbol{u} \right) \mathrm{d}\boldsymbol{x}\mathrm{d}t \\
&\le \int_{Q_{t_F}} \left(\frac{C_{1,\mathrm{w}}}{\rho_{\mathrm{w}}} p_{\mathrm{w}} + (C_{1,\mathrm{w}} C_{4,\mathrm{w}} + C_{1,\mathrm{nw}} C_{4,\mathrm{nw}})|T|^2 + \frac{C_{1,\mathrm{nw}}}{\rho_{\mathrm{nw}}} p_{\mathrm{nw}} \right) \mathrm{d}\boldsymbol{x}\mathrm{d}t \\
&+ \int_{Q_{t_F}} C_2 + \frac{1}{2} \left(|\boldsymbol{f}_u|^2 + |\partial_t \boldsymbol{u}|^2 \right) \mathrm{d}\boldsymbol{x}\mathrm{d}t \\
&\le \int_{Q_{t_F}} \left(\left(\frac{C_{1,\mathrm{w}}}{\rho_{\mathrm{w}}} + \frac{C_{1,\mathrm{nw}}}{\rho_{\mathrm{nw}}} \right) p_{\mathrm{w}} + (C_{1,\mathrm{w}} C_{4,\mathrm{w}} + C_{1,\mathrm{nw}} C_{4,\mathrm{nw}})|T|^2 + \frac{C_{1,\mathrm{nw}}}{\rho_{\mathrm{nw}}} p_{\mathrm{c}} \right) \mathrm{d}\boldsymbol{x}\mathrm{d}t \\
&+ \int_{Q_{t_F}} C_2 + \frac{1}{2} \left(|\boldsymbol{f}_u|^2 + |\partial_t \boldsymbol{u}|^2 \right) \mathrm{d}\boldsymbol{x}\mathrm{d}t
\end{aligned} \tag{3.19}$$

Furthermore, from the definition of the equivalent pore pressure π in (2.7), we have

$$\pi = s_{\mathrm{w}} p_{\mathrm{w}} + s_{\mathrm{nw}} p_{\mathrm{nw}} - U(s_{\mathrm{w}}) = p_{\mathrm{w}} + p_{\mathrm{c}}(s_{\mathrm{w}}) - s_{\mathrm{w}} p_{\mathrm{c}}(s_{\mathrm{w}}) - U(s_{\mathrm{w}}). \tag{3.20}$$

Moreover, differentiating $p_{\mathrm{c}}(s_{\mathrm{w}}) - s_{\mathrm{w}} p_{\mathrm{c}}(s_{\mathrm{w}}) - U(s_{\mathrm{w}})$ with respect to s_{w} and using Assumption (**A7**), we obtain

$$(p_{\mathrm{c}}(s_{\mathrm{w}}) - s_{\mathrm{w}} p_{\mathrm{c}}(s_{\mathrm{w}}) - U(s_{\mathrm{w}}))' = (1 - s_{\mathrm{w}}) p_{\mathrm{c}}'(s_{\mathrm{w}}) \le 0.$$

Then, we have that the function $(1 - s_{\mathrm{w}}) p_{\mathrm{c}}(s_{\mathrm{w}}) - U(s_{\mathrm{w}})$ is decreasing. Since $((1 - s_{\mathrm{w}}) p_{\mathrm{c}} - U)(1) = 0$. We obtain that $(1 - sw) p_{\mathrm{c}}(s_{\mathrm{w}}) - U(s_{\mathrm{w}}) \ge 0$ for all $s_{\mathrm{w}} \in [0, 1]$. Consequently, from (3.20) we obtain that $\pi \ge p_{\mathrm{w}}$. In addition, we have

$$\pi = p^\star - U(s_{\mathrm{w}}) = s_{\mathrm{w}} p_{\mathrm{w}} + s_{\mathrm{nw}} p_{\mathrm{nw}} - U(s_{\mathrm{w}}) = p_{\mathrm{nw}} - s_{\mathrm{w}} p_{\mathrm{c}}(s_{\mathrm{w}}) - U(s_{\mathrm{w}}). \tag{3.21}$$

Using the fact that $U \ge 0$ together with Assumption (**A7**), from (3.21) we obtain that $\pi \le p_{\mathrm{nw}}$. Hence, from equation (3.19), we get

$$\begin{aligned}
&\int_{Q_{t_F}} \left(r_{\mathrm{w}} \left(\frac{p_{\mathrm{w}}}{\rho_{\mathrm{w}}} + e_{\mathrm{w}} - c_{\mathrm{w}} T \right) + r_{\mathrm{nw}} \left(\frac{p_{\mathrm{nw}}}{\rho_{\mathrm{nw}}} + e_{\mathrm{nw}} - c_{\mathrm{nw}} T \right) + r_e + \boldsymbol{f}_u \cdot \partial_t \boldsymbol{u} \right) \mathrm{d}\boldsymbol{x}\mathrm{d}t \\
&\le \int_{Q_{t_F}} \left(\left(\frac{C_{1,\mathrm{w}}}{\rho_{\mathrm{w}}} + \frac{C_{1,\mathrm{nw}}}{\rho_{\mathrm{nw}}} \right) \pi + (C_{1,\mathrm{w}} C_{4,\mathrm{w}} + C_{1,\mathrm{nw}} C_{4,\mathrm{nw}})|T|^2 + \frac{C_{1,\mathrm{nw}}}{\rho_{\mathrm{nw}}} p_{\mathrm{c}} \right) \mathrm{d}\boldsymbol{x}\mathrm{d}t \\
&+ \int_{Q_{t_F}} C_2 + \frac{1}{2} \left(|\boldsymbol{f}_u|^2 + |\partial_t \boldsymbol{u}|^2 \right) \mathrm{d}\boldsymbol{x}\mathrm{d}t \\
&\le \int_{Q_{t_F}} \left(C_7 \left(|\pi|^2 + |T|^2 \right) + C_8 + \frac{1}{2} |\partial_t \boldsymbol{u}|^2 \right) \mathrm{d}\boldsymbol{x}\mathrm{d}t,
\end{aligned} \tag{3.22}$$

where $C_7 := \max \left(\frac{C_{1,\mathrm{w}}}{2\rho_{\mathrm{w}}} + \frac{C_{1,\mathrm{nw}}}{2\rho_{\mathrm{nw}}}, C_{1,\mathrm{w}} C_{4,\mathrm{w}} + C_{1,\mathrm{nw}} C_{4,\mathrm{nw}} \right)$ and $C_8 := C_2 + \frac{C_{1,\mathrm{w}}}{2\rho_{\mathrm{w}}} + \frac{C_{1,\mathrm{nw}}}{2\rho_{\mathrm{nw}}} + \frac{C_{1,\mathrm{nw}}}{\rho_{\mathrm{nw}}} p_{\mathrm{c}}(0) + \frac{1}{2} C_3^2$ are positive constants.

Furthermore, combining the bounds (3.16) and (3.22), equation (3.17) reads

$$\begin{aligned}
\int_0^{t_F} y'(t)\,\mathrm{d}t &\le \int_{Q_{t_F}} \left(C_7 (|\pi|^2 + |T|^2) + C_8 + \frac{1}{2} |\partial_t \boldsymbol{u}|^2 \right) \mathrm{d}\boldsymbol{x}\mathrm{d}t \\
&\le \int_0^{t_F} \left(k_1 y(t) + k_2 \right) \mathrm{d}t,
\end{aligned} \tag{3.23}$$

where $k_1 := \max\left(\frac{C_7}{C_0}, \frac{1}{m_0}\right)$ and $k_2 := C_8|\Omega|$.

From (3.23), we obtain for all $t \in [0, t_F]$

$$y(t) \leq y(0) + \int_0^t (k_1 y(s) + k_2(s))\, ds.$$

As a result of Gronwall's inequality, there exists a constant $C \in \mathbb{R}^+$ such that for all $t \in (0, t_F)$

$$\int_\Omega E(t, \boldsymbol{x})\, \mathrm{d}\boldsymbol{x} + \int_\Omega |\partial_t \boldsymbol{u}(t, \boldsymbol{x})|^2\, \mathrm{d}\boldsymbol{x} \leq C.$$

Therefrom, $E \in L^\infty(0, t_F; L^1(\Omega))$ and $\partial_t \boldsymbol{u} \in L^\infty(0, t_F; (L^2(\Omega))^d)$. From the definition of E, we find that $\phi U \in L^\infty(0, t_F; L^1(\Omega))$ and by Assumption (**A1**), we have that $\phi \geq \phi_*$, then $U \in L^\infty(0, t_F; L^1(\Omega))$.

Furthermore, we have $C_0(|\pi|^2 + |T|^2 + s_{\mathrm{w}} e_{\mathrm{w}} + s_{\mathrm{nw}} e_{\mathrm{nw}}) \leq E$. Since, $s_\alpha \in [0, 1]$ for $\alpha \in \{\mathrm{w}, \mathrm{nw}\}$, it follows that $\pi \in L^\infty(0, t_F; L^2(\Omega))$, $T \in L^\infty(0, t_F; L^2(\Omega))$, $e_{\mathrm{w}} \in L^\infty(0, t_F; L^1(\Omega))$, and $e_{\mathrm{nw}} \in L^\infty(0, t_F; L^1(\Omega))$. Furthermore, since $p_{\mathrm{w}} \leq \pi \leq p_{\mathrm{nw}}$, then $\pi - p_{\mathrm{c}} \leq p_{\mathrm{w}} \leq \pi$ and we conclude that $p_{\mathrm{w}} \in L^\infty(0, t_F; L^2(\Omega))$ and $p_{\mathrm{nw}} \in L^\infty(0, t_F; L^2(\Omega))$. Additionally, by the definition of the total energy E, we have that $\epsilon(\boldsymbol{u}) \in L^\infty(0, t_F; (L^2(\Omega))^{d\times d})$; therefore, $\boldsymbol{\nabla} \boldsymbol{u} \in L^\infty(0, t_F; (L^2(\Omega)^{d\times d}))$, and it follows that $\boldsymbol{u} \in L^\infty(0, t_F; (H_0^1(\Omega))^d)$. Furthermore, using equations (3.17) and (3.22), it follows that

$$\int_{Q_{t_F}} (M_{\mathrm{w}}(s_{\mathrm{w}})\mathbb{K}\boldsymbol{\nabla} p_{\mathrm{w}} \cdot \boldsymbol{\nabla} p_{\mathrm{w}} + M_{\mathrm{nw}}(s_{\mathrm{nw}})\mathbb{K}\boldsymbol{\nabla} p_{\mathrm{nw}} \cdot \boldsymbol{\nabla} p_{\mathrm{nw}})\, \mathrm{d}\boldsymbol{x}\mathrm{d}t \leq C,$$

$$\int_{Q_{t_F}} \frac{\lambda}{T_{\mathrm{ref}}} |\boldsymbol{\nabla} T|^2\, \mathrm{d}\boldsymbol{x}\mathrm{d}t \leq C.$$

Employing equality (2.13), we get

$$\int_{Q_{t_F}} M(s_{\mathrm{w}})|\boldsymbol{\nabla} p|^2 + |\boldsymbol{\nabla} \mathcal{B}(s_{\mathrm{w}})|^2\, \mathrm{d}\boldsymbol{x}\mathrm{d}t \leq C.$$

Then, using Assumption (**A6**), we get that $\boldsymbol{\nabla} p \in L^2(0, t_F; (L^2(\Omega))^d)$, $\boldsymbol{\nabla} \mathcal{B}(s_{\mathrm{w}}) \in L^2(0, t_F; (L^2(\Omega))^d)$, and $\boldsymbol{\nabla} T \in L^2(0, t_F; (L^2(\Omega))^d)$. Finally, we get the desired result. □

4 Weak Solution

We specify the notion of a weak solution of the THM model (2.1). The weak formulation is obtained by multiplying each equation by a separate test function and integrating over space and time. In Section 5, we will construct a numerical scheme for which we establish the convergence to the following weak solution.

Definition 1 (Weak Solution)**.** *Assume that the assumptions in Section 2.2 are fulfilled and suppose that* $(p_{\mathrm{w}}^0, p_{\mathrm{nw}}^0, \boldsymbol{u}^0, T^0) \in (L^2(\Omega))^{d+3}$*, then* $(p_{\mathrm{w}}, p_{\mathrm{nw}}, \boldsymbol{u}, T)$ *is a weak solution of the model satisfying* $p_\alpha \in L^2(0, t_F; L^2(\Omega))$, $\sqrt{M_\alpha(s_\alpha)}\boldsymbol{\nabla} p_\alpha \in L^2(0, t_F; (L^2(\Omega))^d)$ *for any* $\alpha \in \{\mathrm{w}, \mathrm{nw}\}$, $T \in L^2(0, t_F; H^1(\Omega))$, $\boldsymbol{u} \in L^2(0, t_F; (H^1(\Omega))^d)$ *and* $\partial_t \boldsymbol{u} \in L^2(0, t_F; (L^2(\Omega))^d)$,
such that for all $\varphi_\alpha, \psi \in C_c^\infty([0, t_F) \times \overline{\Omega})$ *and for all smooth functions* $\mathbf{v} : [0, t_F] \times \Omega \to \mathbb{R}^d$ *vanishing on* $\partial\Omega$,

$$\begin{aligned} &-\int_{Q_{t_F}} s_{\mathrm{w}}\rho_{\mathrm{w}}\phi\partial_t\varphi_{\mathrm{w}}\, \mathrm{d}\boldsymbol{x}\mathrm{d}t - \int_\Omega s_{\mathrm{w}}^0\rho_{\mathrm{w}}\phi^0\varphi_{\mathrm{w}}^0(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x} \\ &+\int_{Q_{t_F}} M_{\mathrm{w}}(s_{\mathrm{w}})\mathbb{K}\boldsymbol{\nabla} p_{\mathrm{w}} \cdot \boldsymbol{\nabla}\varphi_{\mathrm{w}}\, \mathrm{d}\boldsymbol{x}\mathrm{d}t = \int_{Q_{t_F}} r_{\mathrm{w}}\varphi_{\mathrm{w}}\, \mathrm{d}\boldsymbol{x}\mathrm{d}t, \end{aligned} \tag{4.1}$$

$$\begin{aligned} &-\int_{Q_{t_F}} s_{\mathrm{nw}}\rho_{\mathrm{nw}}\phi\partial_t\varphi_{\mathrm{nw}}\, \mathrm{d}\boldsymbol{x}\mathrm{d}t - \int_\Omega s_{\mathrm{nw}}^0\rho_{\mathrm{nw}}\phi^0\varphi_{\mathrm{nw}}^0(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x} \\ &+\int_{Q_{t_F}} M_{\mathrm{nw}}(s_{\mathrm{nw}})\mathbb{K}\boldsymbol{\nabla} p_{\mathrm{nw}} \cdot \boldsymbol{\nabla}\varphi_{\mathrm{nw}}\, \mathrm{d}\boldsymbol{x}\mathrm{d}t = \int_{Q_{t_F}} r_{\mathrm{nw}}\varphi_{\mathrm{nw}}\, \mathrm{d}\boldsymbol{x}\mathrm{d}t, \end{aligned} \tag{4.2}$$

$$
\begin{aligned}
&-\int_{Q_{t_F}} \left(S_s + \sum_{\alpha\in\{\mathrm{w},\mathrm{nw}\}} \rho_\alpha s_\alpha c_\alpha \phi \right) \partial_t \psi \,\mathrm{d}\boldsymbol{x}\mathrm{d}t \\
&-\int_{\Omega} \left(S_s^0 + \sum_{\alpha\in\{\mathrm{w},\mathrm{nw}\}} \rho_\alpha s_\alpha^0 c_\alpha^0 \phi^0 \right) \psi^0(\boldsymbol{x}) \,\mathrm{d}\boldsymbol{x} + \int_{Q_{t_F}} \frac{\lambda}{T_{\mathrm{ref}}} \boldsymbol{\nabla} T \cdot \boldsymbol{\nabla}\psi \,\mathrm{d}\boldsymbol{x}\mathrm{d}t \qquad (4.3)\\
&+\int_{Q_{t_F}} \sum_{\alpha\in\{\mathrm{w},\mathrm{nw}\}} \rho_\alpha c_\alpha M_\alpha(s_\alpha)\mathbb{K}\boldsymbol{\nabla} p_\alpha \cdot \boldsymbol{\nabla}\psi \,\mathrm{d}\boldsymbol{x}\mathrm{d}t = \int_{Q_{t_F}} \frac{r_e \psi}{T} \,\mathrm{d}\boldsymbol{x}\mathrm{d}t,
\end{aligned}
$$

$$
\begin{aligned}
&-\int_{Q_{t_F}} m_0 \partial_t \boldsymbol{u} \cdot \partial_t \mathbf{v} \,\mathrm{d}\boldsymbol{x}\mathrm{d}t - \int_{\Omega} m_0 \partial_t \boldsymbol{u}^0(\boldsymbol{x}) \cdot \mathbf{v}^0(\boldsymbol{x}) \,\mathrm{d}\boldsymbol{x} \\
&+\int_{Q_{t_F}} \left(\sigma^e(\boldsymbol{u}) : \epsilon(\mathbf{v}) - b\pi \mathrm{div}\,(\mathbf{v}) - 3\alpha_s K_s (T - T_{\mathrm{ref}}) \mathrm{div}\,(\mathbf{v}) \right) \mathrm{d}\boldsymbol{x}\mathrm{d}t \qquad (4.4)\\
&= \int_{Q_{t_F}} \boldsymbol{f}_u \cdot \mathbf{v} \,\mathrm{d}\boldsymbol{x}\mathrm{d}t.
\end{aligned}
$$

5 The finite volume method

In this section, we present the discretization of our model. Consequently, we employ the backward Euler scheme in time and the cell-centered two-point flux approximation (TPFA) finite volume scheme in space following [28] and [38]. In the sequel, we consider an isotropic and homogeneous porous medium; for that, we suppose $\mathbb{K} = k\mathbb{I}_d$, where k is a positive constant and $\mathbb{I}_d \in \mathbb{R}^{d,d}$ is the identity matrix. We also neglect the gravity term for the sake of simplicity.

5.1 Space and time discretizations

For the time discretization, we employ the implicit Euler scheme. For this purpose, we consider an increasing sequence of points $(t^n)_{0\le n\le N_T}$ such that $t^0 := 0$ and $t^{N_T} := t_F$, and we introduce the interval $I_n := (t^{n-1}, t^n)$ and the time step $\tau^n := t^n - t^{n-1}$, $1 \le n \le N_T$. For the sake of simplicity, we assume the time step to be constant so that $\tau^n = \delta t > 0$, $\forall n \in [0, N_T]$. For a function of time f with sufficient regularity, we denote $f^n := f(t^n)$, $0 \le n \le N_T$, and for $1 \le n \le N_T$, we define the backward differencing operator

$$
\partial_t^n f := \frac{f^n - f^{n-1}}{\delta t}.
$$

For the space discretization of our model, we consider $\mathcal{T}$ an admissible orthogonal mesh of Ω called primal mesh such that $\overline{\Omega} = \bigcup_{K\in\mathcal{T}} \overline{K}$ where K are open and convex polygons called control volumes. We denote by $\mathcal{E}_h$ the set of mesh edges. Boundary edges are collected in the set $\mathcal{E}_h^{\mathrm{ext}} := \{\sigma \in \mathcal{E}_h;\, \sigma \subset \partial\Omega\}$ and internal edges are collected in the set $\mathcal{E}_h^{\mathrm{int}} = \mathcal{E}_h \setminus \mathcal{E}_h^{\mathrm{ext}}$. Likewise, the edges of an element $K \in \mathcal{T}$ are collected in the set $\mathcal{E}_K$, and the latter is decomposed into interior edges $\mathcal{E}_K^{\mathrm{int}}$ and boundary edges $\mathcal{E}_K^{\mathrm{ext}}$. For all $K \in \mathcal{T}$, we denote by $\boldsymbol{x}_K$ the cell center of K and $\mathcal{N}(K)$ the set of its neighbors defined as

$$
\mathcal{N}(K) := \{L \in \mathcal{T}; \exists\, \sigma_{KL} \in \mathcal{E}_K,\, \sigma_{KL} = \overline{K} \cap \overline{L}\} = \mathcal{N}_{\mathrm{int}}(K) \cup \mathcal{N}_{\mathrm{ext}}(K),
$$

where $\mathcal{N}_{\mathrm{int}}(K)$ is the set of neighbors of K located in the interior of $\mathcal{T}$ and $\mathcal{N}_{\mathrm{ext}}(K)$ is the set of edges of K on the boundary $\partial\Omega = \partial K \cap \partial\Omega$. For an edge $\sigma_{KL} \in \mathcal{E}_K$ shared by two elements K and L, we define the distance between these elements $d_{KL} := \mathrm{dist}(\boldsymbol{x}_K, \boldsymbol{x}_L)$ and $\tau_{KL} := \dfrac{|\sigma_{KL}|}{d_{KL}}$ as the transmissibility coefficient through σ_{KL}.

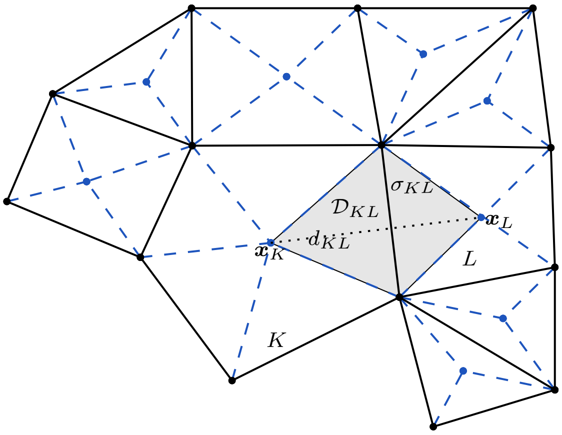

Figure 1: Illustration of the primal and dual meshes. For two elements K and L shared by the edge σ_{KL}, the corresponding diamond in gray is $\mathcal{D}_{KL}$

To discretize our unknowns, we introduce the discrete space of cellwise constant functions as follows

$$L_h := \left\{ v_h \in L^2(\Omega),\, v_h|_K = v_K \,\forall K \in \mathcal{T} \right\}.$$

We also define the scalar product on the space L_h and its underlying norm by

$$(w_h, v_h)_{L^2(\Omega)} := \sum_{K\in\mathcal{T}} |K| w_K v_K, \quad \text{and} \quad \|v_h\|_{L^2(\Omega)} := \left(\sum_{K\in\mathcal{T}} |K| |v_K|^2 \right)^{\frac{1}{2}}.$$

Moreover, we construct the dual diamond cell upon the interface σ_{KL} as the polygonal domain having $\boldsymbol{x}_K$ and $\boldsymbol{x}_L$ as vertices. We denote it by $\mathcal{D}_{KL}$. The d-dimensional measure $|\mathcal{D}_{KL}|$ of $\mathcal{D}_{KL}$ is given by

$$|\mathcal{D}_{KL}| := \frac{1}{d} |\sigma_{KL}| d_{KL}.$$

For two elements K and L shared by an interface σ_{KL}, we associate a diamond $\mathcal{D}_{KL}$. For an external edge $\sigma \subset \partial\Omega$, the corresponding diamond is a half diamond denoted by $\mathcal{D}_{K\sigma}$. We also denote

$$\mathcal{D}_{\text{int}} := \{\mathcal{D}_{KL},\ K \in \mathcal{T} \text{ and } L \in \mathcal{N}(K)\}, \quad \mathcal{D}_{\text{ext}} := \{\mathcal{D}_{K\sigma}, K \in \mathcal{T} \text{and}\, \sigma \in \mathcal{E}_K^{\text{ext}}\}.$$

Then, we have

$$\overline{\Omega} = \mathcal{D}_{\text{int}} \cup \mathcal{D}_{\text{ext}},$$

and the mesh composed of the diamonds is called the dual mesh.

For a piecewise constant function $v_h \in L_h$ defined per control volume, we define on the dual mesh the discrete broken gradient $\boldsymbol{\nabla}_h v_h \in \mathbb{R}^d$ as a constant per diamond by

$$\boldsymbol{\nabla}_h v_h(\boldsymbol{x}) := \begin{cases} d\frac{v_L - v_K}{d_{KL}} \boldsymbol{n}_{KL} & \text{if } \boldsymbol{x} \in \mathcal{D}_{KL}, \\ d\frac{v_\sigma - v_K}{d_{K\sigma}} \boldsymbol{n}_{K\sigma} & \text{if } \boldsymbol{x} \in \mathcal{D}_{K\sigma}. \end{cases}$$

Analogously, for a piecewise constant vector function $\boldsymbol{w}_h \in L_h^d$ defined per control volume, we define on the dual mesh the discrete broken gradient $\boldsymbol{\nabla}_h \boldsymbol{w}_h \in \mathbb{R}^{d\times d}$ by

$$\boldsymbol{\nabla}_h \boldsymbol{w}_h(\boldsymbol{x}) := \begin{cases} d\frac{\boldsymbol{w}_L - \boldsymbol{w}_K}{d_{KL}} \otimes \boldsymbol{n}_{KL} & \text{if } \boldsymbol{x} \in \mathcal{D}_{KL}, \\ d\frac{\boldsymbol{w}_\sigma - \boldsymbol{w}_K}{d_{K\sigma}} \otimes \boldsymbol{n}_{K\sigma} & \text{if } \boldsymbol{x} \in \mathcal{D}_{K\sigma}. \end{cases}$$

When there is no confusion, we write $\sum_{\sigma_{KL}\in\mathcal{E}_h}$ to be the sum over all diamonds, including the diamonds associated with interfaces on the boundary of the domain.

Furthermore, for $(w_h, v_h) \in (L_h)^2$, we define the discrete H_0^1 inner product as follows:

$$\langle w_h, v_h \rangle_{H_h(\Omega)} := d \sum_{\sigma_{KL} \in \mathcal{E}_h} \tau_{KL}(w_L - w_K)(v_L - v_K),$$

and the corresponding discrete H_0^1 norm as $\|v_h\|_{H_h(\Omega)} := \left(\langle v_h, v_h \rangle_{H_h(\Omega)}\right)^{1/2} = \left(d \sum_{\sigma_{KL} \in \mathcal{E}_h} \tau_{KL}(v_L - v_K)^2\right)^{1/2}$. In addition, the norm $\|v_h\|_{H_h}$ coincides with the $L^2(\Omega)$ norm of $\boldsymbol{\nabla}_h v_h$. In fact

$$\begin{aligned} \|\boldsymbol{\nabla}_h v_h\|^2_{L^2(\Omega)} &= \int_\Omega |\boldsymbol{\nabla}_h v_h|^2 \, \mathrm{d}\boldsymbol{x} = \sum_{\sigma_{KL} \in \mathcal{E}_h} \int_{\mathcal{D}_{KL}} |\boldsymbol{\nabla}_h v_h|^2 \mathrm{d}\boldsymbol{x} \\ &= d \sum_{\sigma_{KL} \in \mathcal{E}_h} \tau_{KL} |v_L - v_K|^2 := \|v_h\|^2_{H_h}. \end{aligned}$$

Additionally, we assimilate a discrete field $\boldsymbol{G}_{KL}$ to the diamond piecewise constant vector function

$$\boldsymbol{G}_h = \sum_{\sigma_{KL} \in \mathcal{E}_h} \boldsymbol{G}_{KL} \mathbb{1}_{\mathcal{D}_{KL}}.$$

where $\mathbb{1}_{\mathcal{D}_{KL}}$ is the indicator function of the set $\mathcal{D}_{KL}$. Moreover, we define the discrete divergence, piecewise constant on the control volume $K \in \mathcal{T}$, of the vector field $\boldsymbol{G}_h$ by

$$\operatorname{div}_K \boldsymbol{G}_h := \frac{1}{|K|} \sum_{L \in \mathcal{N}(K)} |\sigma_{KL}| \boldsymbol{G}_h \cdot \boldsymbol{n}_{KL}.$$

Finally, we define the finite volume discretization $\mathscr{D}$ of $\Omega \times (0, t_F)$ by

$$\mathscr{D} = \left(\mathcal{T}, \mathcal{E}_h, (\boldsymbol{x}_K)_{K \in \mathcal{T}}, N_T, (t^n)_{n \in [1, N_T]}\right).$$

We also set $\operatorname{size}(\mathscr{D}) = \max(\operatorname{size}(\mathcal{T}), \delta t)$. In addition, we associate a time and space piecewise constant function to the sequence $(v_K^n)_{K \in \mathcal{T}, n \in [1, N_T]}$, denoted by $v_{\mathscr{D}}$ and defined almost everywhere on $\Omega \times (0, t_F)$ by

$$v_{\mathscr{D}}(t, \boldsymbol{x}) = v_K^n, \text{ for a.e } (t, \boldsymbol{x}) \in (t^{n-1}, t^n) \times K, \ \forall K \in \mathcal{T}, \ \forall n \in [1, N_T].$$

5.2 The finite volume scheme

The finite volume scheme is obtained by integrating the model (2.1) on each control volume. First, we discretize the wetting fluid mass conservation equation (2.1a). Let $K \in \mathcal{T}$. By integrating (2.1a) over $[t^{n-1}, t^n] \times K$ and using the divergence theorem, we obtain

$$\begin{aligned} \int_K \rho_{\mathrm{w}} \left((s_{\mathrm{w}}\phi)(t^n, \boldsymbol{x}) - (s_{\mathrm{w}}\phi)(t^{n-1}, x)\right) \mathrm{d}\boldsymbol{x} + \delta t \int_{t^{n-1}}^{t^n} \int_{\partial K} \rho_{\mathrm{w}} \boldsymbol{V}_{\mathrm{w}}(p_{\mathrm{w}}) \cdot \boldsymbol{n} \, d\sigma \mathrm{d}t \\ = \delta t \int_{t^{n-1}}^{t^n} \int_K r_{\mathrm{w}}(t, \boldsymbol{x}) \, \mathrm{d}\boldsymbol{x} \mathrm{d}t. \end{aligned} \tag{5.1}$$

The resulting equation is discretized using the implicit Euler scheme in time, and the normal gradients are discretized with a cell-centered finite difference scheme. Then for $n = 1, \ldots, N_T$, (5.1) reads

$$|K| \rho_{\mathrm{w}} \left(s^n_{\mathrm{w},K} \phi^n_K - s^{n-1}_{\mathrm{w},K} \phi^{n-1}_K\right) + \delta t \sum_{L \in \mathcal{N}(K)} \mathcal{F}_{1,KL}(p^n_{\mathrm{w}}) = \delta t \, r^n_{\mathrm{w},K}. \tag{5.2}$$

Here, the discrete elementwise source term $r^n_{\mathrm{w},K}$ is defined by

$$r^n_{\mathrm{w},K} := \frac{1}{|K| \delta t} \int_{I_n} \int_K r_{\mathrm{w}}(t, \boldsymbol{x}) \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}t.$$

Furthermore, the total flux across $\sigma_{KL} \in \mathcal{E}_h^{\text{int}}$, $\sigma_{KL} = \partial K \cap \partial L$, is defined by

$$\mathcal{F}_{1,KL}(p_{\text{w}}^n) := \rho_{\text{w}} k M_{\text{w},KL}^n \tau_{KL}(p_{\text{w},K}^n - p_{\text{w},L}^n),$$

where $M_{\alpha,KL}^n$ is the value of the mobility of the phase $\alpha \in \{\text{w}, \text{nw}\}$ at the interfaces and it is defined by an upwind choice according to the discrete gradient of its own pressure

$$M_{\alpha,KL}^n := \begin{cases} M_\alpha(s_{\alpha,K}^n) & \text{if } p_{\alpha,K}^n - p_{\alpha,L}^n \geq 0, \\ M_\alpha(s_{\alpha,L}^n) & \text{otherwise.} \end{cases} \tag{5.3}$$

We observe that the numerical flux $\mathcal{F}_{1,KL}(p_{\text{w}}^n)$ is conservative in the sense that $\mathcal{F}_{1,LK}(p_{\text{w}}^n) = -\mathcal{F}_{1,KL}(p_{\text{w}}^n)$. Similarly, we discretize the non-wetting fluid mass conservation equation (2.1b). We obtain

$$|K| \rho_{\text{nw}} \left(s_{\text{nw},K}^n \phi_K^n - s_{\text{nw},K}^{n-1} \phi_K^{n-1} \right) + \delta t \sum_{L \in \mathcal{N}(K)} \mathcal{F}_{2,KL}(p_{\text{nw}}^n) = \delta t \; r_{\text{nw},K}^n. \tag{5.4}$$

Here, the discrete elementwise source term $r_{\text{nw},K}^n$ is defined by

$$r_{\text{nw},K}^n := \frac{1}{|K| \delta t} \int_{I_n} \int_K r_{\text{nw}}(t, \boldsymbol{x}) \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}t.$$

Furthermore, the total flux across $\sigma_{KL} \in \mathcal{E}_h^{\text{int}}$, $\sigma_{KL} = \partial K \cap \partial L$, is defined by

$$\mathcal{F}_{2,KL}(p_{\text{nw}}^n) := \rho_{\text{nw}} k M_{\text{nw},KL}^n \tau_{KL}(p_{\text{nw},K}^n - p_{\text{nw},L}^n),$$

where the non-wetting phase mobility $M_{\text{nw},KL}^n$ is defined by (5.3) at the interfaces. We observe that the numerical flux $\mathcal{F}_{2,KL}(p_{\text{nw}}^n)$ is conservative in the sense that $\mathcal{F}_{2,LK}(p_{\text{nw}}^n) = -\mathcal{F}_{2,KL}(p_{\text{nw}}^n)$. Moreover, we discretize the entropy conservation equation (2.1c). Let $K \in \mathcal{T}$. By integrating (2.1c) over $[t^{n-1}, t^n] \times K$ and using the divergence theorem, we obtain

$$\begin{aligned} &\int_K \left((S_s + \sum_{\alpha \in \{\text{w},\text{nw}\}} \rho_\alpha \phi s_\alpha c_\alpha)(t^n, \boldsymbol{x}) - (S_s + \sum_{\alpha \in \{\text{w},\text{nw}\}} \rho_\alpha \phi s_\alpha c_\alpha)(t^{n-1}, \boldsymbol{x}) \right) dx \\ &+ \int_{t^{n-1}}^{t^n} \int_{\partial K} \sum_{\alpha \in \{\text{w},\text{nw}\}} \rho_\alpha c_\alpha \boldsymbol{V}_\alpha(p_\alpha) \cdot \boldsymbol{n} \, d\sigma dt + \int_{t^{n-1}}^{t^n} \int_{\partial K} \frac{1}{T_{\text{ref}}} \boldsymbol{q}(T) \cdot \boldsymbol{n} \, d\sigma dt \\ &= \int_{t^{n-1}}^{t^n} \int_K \frac{r_e(t, \boldsymbol{x})}{T} \, dx dt. \end{aligned} \tag{5.5}$$

The resulting equation is discretized using the implicit Euler scheme in time, and the normal gradients are discretized with a centered finite difference scheme. Then, for $n = 1, \ldots, N_T$, (5.5) reads

$$\begin{aligned} &|K| \left(S_{s,K}^n + \sum_{\alpha \in \{\text{w},\text{nw}\}} \rho_\alpha \phi_K^n s_{\alpha,K}^n c_{\alpha,K}^n - S_{s,K}^{n-1} - \sum_{\alpha \in \{\text{w},\text{nw}\}} \rho_\alpha \phi_K^{n-1} s_{\alpha,K}^{n-1} c_{\alpha,K}^{n-1} \right) \\ &+ \delta t \sum_{L \in \mathcal{N}(K)} \mathcal{F}_{3,KL}(p_{\text{w}}^n, p_{\text{nw}}^n, T^n) = \delta t |K| \frac{r_{e,K}^n}{T_K^n}. \end{aligned} \tag{5.6}$$

Here, for $L \in \mathcal{N}(K)$, we have

$$\begin{aligned} \mathcal{F}_{3,KL}(p_{\text{w}}^n, p_{\text{nw}}^n, T^n) := &\sum_{\alpha \in \{\text{w},\text{nw}\}} \rho_\alpha c_{\alpha,KL}^n k M_{\alpha,KL}^n \tau_{KL} \left(p_{\alpha,K}^n - p_{\alpha,L}^n \right) \\ &+ \frac{\lambda}{T_{\text{ref}}} \tau_{KL} \left(T_K^n - T_L^n \right), \end{aligned}$$

with

$$c_{\alpha,KL}^n := \frac{c_{\alpha,K}^n + c_{\alpha,L}^n}{2}, \quad \text{and } M_{\alpha,KL}^n \text{ defined by (5.3).}$$

Furthermore, the discrete elementwise source term $r^n_{e,K}$ is defined by

$$r^n_{e,K} := \frac{1}{|K|\delta t}\int_{I_n}\int_K r_e(t,\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t.$$

Observe that $\mathcal{F}_{3,LK}(p^n_{\mathrm{w}}, p^n_{\mathrm{nw}}, T^n) = -\mathcal{F}_{3,KL}(p^n_{\mathrm{w}}, p^n_{\mathrm{nw}}, T^n)$, which provides a conservative numerical approximation.

Finally, we discretize the momentum conservation equation (2.1d). Let $K \in \mathcal{T}$. By integrating (2.1d) over $[t^{n-1}, t^n] \times K$ and using the divergence theorem, we obtain

$$\begin{aligned}\int_K m_0\left(\partial_t \boldsymbol{u}(t^n,\boldsymbol{x}) - \partial_t \boldsymbol{u}(t^{n-1},\boldsymbol{x})\right)\,dx - \int_{t^{n-1}}^{t^n}\int_{\partial K} \boldsymbol{\sigma}(\boldsymbol{u}, p_{\mathrm{w}}, p_{\mathrm{nw}}, T)\boldsymbol{n}\,d\sigma dt \\ = \int_{t^{n-1}}^{t^n}\int_K \boldsymbol{f}_u(t,\boldsymbol{x})\,dx\mathrm{d}t.\end{aligned} \tag{5.7}$$

From the definition of the total-stress tensor (2.6), equation (5.7) reads

$$\begin{aligned}&\int_K m_0\left(\partial_t \boldsymbol{u}(t^n,\boldsymbol{x}) - \partial_t \boldsymbol{u}(t^{n-1},\boldsymbol{x})\right)\,dx \\ &- \int_{t^{n-1}}^{t^n}\int_{\partial K} \frac{\mathcal{E}}{1+\nu}\left(\boldsymbol{\epsilon}(\boldsymbol{u}) + \frac{\nu}{1-2\nu}\operatorname{div}\boldsymbol{u}\mathbb{I}_d\right)\boldsymbol{n}\,d\sigma \\ &+ \int_{t^{n-1}}^{t^n}\int_{\partial K} \left(b\pi\mathbb{I}_d + 3\alpha_s K_s(T - T_{\mathrm{ref}})\mathbb{I}_d\right)\boldsymbol{n}\,d\sigma = \int_{t^{n-1}}^{t^n}\int_K \boldsymbol{f}_u(t,\boldsymbol{x})\,dx\mathrm{d}t.\end{aligned}$$

Then, using the backward differencing operator, we get

$$|K|\frac{\boldsymbol{u}^n_K + \boldsymbol{u}^{n-2}_K - 2\boldsymbol{u}^{n-1}_K}{\delta t} + \delta t\sum_{L\in\mathcal{N}(K)} \mathcal{F}^n_{4,KL}(\boldsymbol{u}^n, \pi^n, T^n) = \delta t|K|\boldsymbol{f}^n_{u,K}. \tag{5.8}$$

For $L \in \mathcal{N}(K)$, the total flux $\mathcal{F}_{4,KL}(\boldsymbol{u}^n, \pi^n, T^n)$ is defined by

$$\begin{aligned}&\mathcal{F}_{4,KL}(\boldsymbol{u}^n, \pi^n, T^n) := \\ &\frac{\mathcal{E}}{(1+\nu)}\frac{\tau_{KL}}{2}\left[\left((\boldsymbol{u}^n_K - \boldsymbol{u}^n_L)\otimes\boldsymbol{n}_{KL}\right)\boldsymbol{n}_{KL} + \left[(\boldsymbol{u}^n_K - \boldsymbol{u}^n_L)\otimes\boldsymbol{n}_{KL}\right]^\top\boldsymbol{n}_{KL}\right] \\ &- \frac{\mathcal{E}\nu}{(1+\nu)(1-2\nu)}|\sigma_{KL}|\left(\operatorname{div}\boldsymbol{u}^n\mathbb{I}_d\right)_{KL}\boldsymbol{n}_{KL} + b|\sigma_{KL}|\pi^n_{KL}\mathbb{I}_d\boldsymbol{n}_{KL} \\ &+ |\sigma_{KL}|3\alpha_s K_s\left(T^n_{KL} - T_{\mathrm{ref}}\right)\mathbb{I}_d\boldsymbol{n}_{KL},\end{aligned}$$

where

$$(\operatorname{div}\boldsymbol{u}^n\mathbb{I}_d)_{KL} := \frac{\operatorname{div}_K\boldsymbol{u}^n_{KL} + \operatorname{div}_L\boldsymbol{u}^n_{KL}}{2}\mathbb{I}_d, \quad \pi^n_{KL} := \frac{\pi^n_K + \pi^n_L}{2}, \quad T^n_{KL} := \frac{T^n_K + T^n_L}{2}.$$

Furthermore, the discrete elementwise source term $\boldsymbol{f}^n_{u,K}$ is defined by

$$\boldsymbol{f}^n_{u,K} := \frac{1}{|K|\delta t}\int_{I_n}\int_K \boldsymbol{f}_u(t,\boldsymbol{x})\,dx\,dt.$$

We observe that $\mathcal{F}^n_{4,KL}(\boldsymbol{u}^n, \pi^n, T^n) = -\mathcal{F}^n_{4,LK}(\boldsymbol{u}^n, \pi^n, T^n)$, which provides a conservative numerical approximation. By integrating the porosity and skeleton entropy equations (2.9) over $[t^{n-1}, t^n] \times K$ and using the divergence theorem together with the discrete divergence definition, we obtain the following discrete porosity $(\phi^n_h)_{n=1,\ldots,N_T}$ and skeleton entropy $(S^n_{s,h})_{n=1,\ldots,N_T}$

$$\begin{aligned}\frac{\phi^n_K - \phi^{n-1}_K}{\delta t} &= b\operatorname{div}_K\partial^n_t\boldsymbol{u} - 3\alpha_\phi\frac{T^n_K - T^{n-1}_K}{\delta t} + \frac{1}{N}\frac{\pi^n_K - \pi^{n-1}_K}{\delta t}, \\ \frac{S^n_{s,K} - S^{n-1}_{s,K}}{\delta t} &= 3\alpha_s K_s\operatorname{div}_K\partial^n_t\boldsymbol{u} - 3\alpha_\phi\frac{\pi^n_K - \pi^{n-1}_K}{\delta t} + \frac{C_s}{T_{\mathrm{ref}}}\frac{T^n_K - T^{n-1}_K}{\delta t}.\end{aligned} \tag{5.9}$$

Furthermore, we discretize the energy-entropy relation (2.10). Let $K \in \mathcal{T}$ and $\alpha \in \{\mathrm{w}, \mathrm{nw}\}$. By integrating the first equation of (2.10) over $[t^{n-1}, t^n] \times K$, we obtain

$$c^n_{\alpha,K} - c^{n-1}_{\alpha,K} = \frac{1}{T^n_K}\left(e^n_{\alpha,K} - e^{n-1}_{\alpha,K}\right). \tag{5.10}$$

Let $\psi \in H^1(\Omega)$ be a test function. Multiplying the second equation of (2.10) by (ψM_α) and applying the divergence operator to both sides, we obtain:

$$\operatorname{div}\left(\psi M_\alpha \boldsymbol{\nabla} c_\alpha\right) = \operatorname{div}\left(\frac{\psi M_\alpha}{T}\boldsymbol{\nabla} e_\alpha\right). \tag{5.11}$$

Let $K \in \mathcal{T}$. By integrating the equation (5.11) over $[t^{n-1}, t^n] \times K$ and using the divergence theorem, we obtain

$$\sum_{L\in\mathcal{N}(K)} \psi_{KL} M^n_{\alpha,KL} \tau_{KL} (c^n_{\alpha,L} - c^n_{\alpha,K}) = \sum_{L\in\mathcal{N}(K)} \frac{\tau_{KL}\psi_{KL}M^n_{\alpha,KL}}{T^n_{KL}}(e^n_{\alpha,L} - e^n_{\alpha,K}), \tag{5.12}$$

where $T^n_{KL} := \dfrac{T^n_K + T^n_L}{2}$, $\psi_{KL} := \dfrac{\psi_K + \psi_L}{2}$, and $M^n_{\alpha,KL}$ is defined by (5.3).

Moreover, we obtain from the definition of the equivalent pore pressure (2.7), the following discrete relation

$$\pi^n_h = p^{\star,n}_h - U^n_h, \text{ with } p^{\star,n}_h = s^n_{\mathrm{w},h} p^n_{\mathrm{w},h} + s^n_{\mathrm{nw},h} p^n_{\mathrm{nw},h} \text{ and } U^n_h = U(s^n_{\mathrm{w},h}) = \int_{s^n_{\mathrm{w},h}}^1 p_{\mathrm{c}}(z)\,\mathrm{d}z. \tag{5.13}$$

Finally, at every time step $1 \le n \le N_T$, we are looking for $\mathcal{U}^n_h = (\boldsymbol{u}^n_h, p^n_{\mathrm{w},h}, p^n_{\mathrm{nw},h}, T^n_h) \in \mathbb{R}^{(3+d)|\mathcal{T}_h|}$ a solution to the nonlinear system of algebraic equations (5.2), (5.4), (5.6) and (5.8). We aim to show that this scheme is convergent to a weak solution in the sense of Definition 1. To this end, we first show that the finite volume scheme (5.2)–(5.8) can be written as a discrete variational formulation. Secondly, we show that the energy estimates obtained in Section 3 are well-preserved at the discrete level. Finally, we establish some compactness properties to pass to the limit.

5.3 Discrete variational formulation

In this section, we aim to show that the finite volume scheme (5.2)–(5.8) is equivalent to a discrete variational formulation. To do so, we state first a key identity and a reformulation to the discrete entropy conservation equation (5.6).

Lemma 2. *Let $\psi_h \in H_h$. We have the identity:*

$$\begin{aligned}&\sum_{K\in\mathcal{T}}\sum_{L\in\mathcal{N}(K)}\sum_{\alpha\in\{\mathrm{w},\mathrm{nw}\}} \rho_\alpha k M^n_{\alpha,KL}(c^n_{\alpha,L} - c^n_{\alpha,K})\tau_{KL}\left(p^n_{\alpha,L} - p^n_{\alpha,K}\right)\psi_K \\ &= \sum_{K\in\mathcal{T}}\sum_{L\in\mathcal{N}(K)}\sum_{\alpha\in\{\mathrm{w},\mathrm{nw}\}} \frac{\rho_\alpha k M^n_{\alpha,KL}\tau_{KL}}{T^n_{KL}}(e^n_{\alpha,L} - e^n_{\alpha,K})(p^n_{\alpha,L} - p^n_{\alpha,K})\psi_{KL}.\end{aligned}$$

Proof of Lemma 2. Since $M_{\alpha,KL}$ is symmetric meaning $M_{\alpha,LK} = M_{\alpha,KL}$ for all $\alpha \in \{\mathrm{w}, \mathrm{nw}\}, K \in \mathcal{T}$ and $L \in \mathcal{N}(K)$, then the proof is similar to the one given in [35, Proposition 2]. □

The following proposition provides a reformulation of the discrete entropy conservation equation (5.6), using the discrete entropy-energy relation (5.10)-(5.12) and the identity stated in Lemma 2.

Proposition 2. *Let $\psi_h \in H_h$ be a test function. The discrete entropy balance equation* (5.6) *is equivalent to*

$$\begin{aligned}
&\sum_{K\in\mathcal{T}} |K| \left(S^n_{s,K} - S^{n-1}_{s,K} + \sum_{\alpha\in\{\mathrm{w},\mathrm{nw}\}} \frac{\rho_\alpha \phi^{n-1}_K s^{n-1}_{\alpha,K}}{T^n_K} \left(e^n_{\alpha,K} - e^{n-1}_{\alpha,K} \right) \right) \psi_K \\
&+ \frac{1}{2}\delta t \sum_{K\in\mathcal{T}} \sum_{L\in\mathcal{N}(K)} \sum_{\alpha\in\{\mathrm{w},\mathrm{nw}\}} \frac{\rho_\alpha k M^n_{\alpha,KL}\tau_{KL}}{T^n_{KL}} (e^n_{\alpha,L} - e^n_{\alpha,K})(p^n_{\alpha,K} - p^n_{\alpha,L})\psi_{KL} \\
&+ \delta t \sum_{K\in\mathcal{T}} \sum_{L\in\mathcal{N}(K)} \frac{\lambda}{T_{\mathrm{ref}}} (T^n_K - T^n_L)\psi_K \\
&= \delta t \sum_{K\in\mathcal{T}} |K| \frac{r^n_{e,K}}{T^n_K}\psi_K - \delta t \sum_{K\in\mathcal{T}} \sum_{\alpha\in\{\mathrm{w},\mathrm{nw}\}} |K| c^n_{\alpha,K} r^n_{\alpha,K}\psi_K.
\end{aligned} \tag{5.14}$$

Proof of Proposition 2. Let $\psi_h \in H_h$ be a test function. Multiplying equation (5.6) by ψ_K and summing over K, we obtain

$$\begin{aligned}
&\sum_{K\in\mathcal{T}} |K| \left(S^n_{s,K} - S^{n-1}_{s,K} + \sum_{\alpha\in\{\mathrm{w},\mathrm{nw}\}} \rho_\alpha \left(\phi^n_K s^n_{\alpha,K} c^n_{\alpha,K} - \phi^{n-1}_K s^{n-1}_{\alpha,K} c^{n-1}_{\alpha,K} \right) \right) \psi_K \\
&+ \delta t \sum_{K\in\mathcal{T}} \sum_{L\in\mathcal{N}(K)} \sum_{\alpha\in\{\mathrm{w},\mathrm{nw}\}} \rho_\alpha c^n_{\alpha,KL} k M^n_{\alpha,KL}\tau_{KL}(p^n_{\alpha,K} - p^n_{\alpha,L})\psi_K \\
&+ \delta t \sum_{K\in\mathcal{T}} \sum_{L\in\mathcal{N}(K)} \sum_{\alpha\in\{\mathrm{w},\mathrm{nw}\}} \frac{\lambda}{T_{\mathrm{ref}}} (T^n_K - T^n_L)\psi_K = \delta t \sum_{K\in\mathcal{T}} |K| \frac{r^n_{e,K}}{T^n_K}\psi_K.
\end{aligned} \tag{5.15}$$

To handle the second term of (5.15), we use the following equality

$$\begin{aligned}
\phi^n_K s^n_{\alpha,K} c^n_{\alpha,K} - \phi^{n-1}_K s^{n-1}_{\alpha,K} c^{n-1}_{\alpha,K} &= \delta t \left(c^n_{\alpha,K} \partial^n_t (s_{\alpha,K}\phi_K) + s^{n-1}_{\alpha,K}\phi^{n-1}_K \partial^n_t c_{\alpha,K} \right) \\
&= \delta t \left(c^n_{\alpha,K} \partial^n_t (s_{\alpha,K}\phi_K) + \frac{s^{n-1}_{\alpha,K}\phi^{n-1}_K}{T^n_K} \partial^n_t e_{\alpha,K} \right).
\end{aligned} \tag{5.16}$$

Moreover, using the definition of $c^n_{\alpha,KL}$, we can rewrite the third term of (5.15) as follows:

$$\begin{aligned}
c^n_{\alpha,KL}(p^n_{\alpha,K} - p^n_{\alpha,L}) &= (c^n_{\alpha,KL} - c^n_{\alpha,K})(p^n_{\alpha,K} - p^n_{\alpha,L}) + c^n_{\alpha,K}(p^n_{\alpha,K} - p^n_{\alpha,L}) \\
&= \frac{1}{2}(c^n_{\alpha,L} - c^n_{\alpha,K})(p^n_{\alpha,K} - p^n_{\alpha,L}) + c^n_{\alpha,K}(p^n_{\alpha,K} - p^n_{\alpha,L}).
\end{aligned} \tag{5.17}$$

Combining equations (5.16) and (5.17), equation (5.15) reads

$$\begin{aligned}
&\sum_{K\in\mathcal{T}} |K| \left(S^n_{s,K} - S^{n-1}_{s,K} \right) \psi_K \\
&+ \sum_{K\in\mathcal{T}} |K| \sum_{\alpha\in\{\mathrm{w},\mathrm{nw}\}} \rho_\alpha \delta t \left(c^n_{\alpha,K} \partial^n_t (s_{\alpha,K}\phi_K) + \frac{s^{n-1}_{\alpha,K}\phi^{n-1}_K}{T^n_K} \partial^n_t e_{\alpha,K} \right) \psi_K \\
&+ \frac{\delta t}{2} \sum_{K\in\mathcal{T}} \sum_{L\in\mathcal{N}(K)} \sum_{\alpha\in\{\mathrm{w},\mathrm{nw}\}} \rho_\alpha (c^n_{\alpha,L} - c^n_{\alpha,K}) k M^n_{\alpha,KL}\tau_{KL}(p^n_{\alpha,K} - p^n_{\alpha,L})\psi_K \\
&+ \delta t \sum_{K\in\mathcal{T}} \sum_{L\in\mathcal{N}(K)} \sum_{\alpha\in\{\mathrm{w},\mathrm{nw}\}} \rho_\alpha c^n_{\alpha,K} k M^n_{\alpha,KL}\tau_{KL}(p^n_{\alpha,K} - p^n_{\alpha,L})\psi_K \\
&+ \delta t \sum_{K\in\mathcal{T}} \sum_{L\in\mathcal{N}(K)} \sum_{\alpha\in\{\mathrm{w},\mathrm{nw}\}} \frac{\lambda}{T_{\mathrm{ref}}} (T^n_K - T^n_L)\psi_K = \delta t \sum_{K\in\mathcal{T}} |K| \frac{r^n_{e,K}}{T^n_K}\psi_K.
\end{aligned} \tag{5.18}$$

Substituting the wetting and non-wetting fluids discrete mass conservation equations (5.4) and (5.2) in (5.18), we obtain

$$
\begin{aligned}
&\sum_{K\in\mathcal{T}} |K| \left(S^n_{s,K} - S^{n-1}_{s,K} + \delta t \sum_{\alpha\in\{\mathrm{w},\mathrm{nw}\}} \rho_\alpha \frac{s^{n-1}_{\alpha,K}\phi^{n-1}_K}{T^n_K} \partial^n_t e_{\alpha,K} \right) \psi_K \\
&+ \frac{\delta t}{2} \sum_{K\in\mathcal{T}} \sum_{L\in\mathcal{N}(K)} \sum_{\alpha\in\{\mathrm{w},\mathrm{nw}\}} \rho_\alpha (c^n_{\alpha,L} - c^n_{\alpha,K}) k M^n_{\alpha,KL} \tau_{KL} (p^n_{\alpha,K} - p^n_{\alpha,L}) \psi_K \\
&+ \delta t \sum_{K\in\mathcal{T}} \sum_{L\in\mathcal{N}(K)} \frac{\lambda}{T_{\mathrm{ref}}} (T^n_K - T^n_L) \psi_K \\
&= \delta t \sum_{K\in\mathcal{T}} |K| \left(\frac{r^n_{e,K}}{T^n_K} - c^n_{\mathrm{w},K} r^n_{\mathrm{w},K} - c^n_{\mathrm{nw},K} r^n_{\mathrm{nw},K} \right) \psi_K .
\end{aligned}
\tag{5.19}
$$

Finally, by applying Lemma 2 to the third term of (5.19), we obtain equation (5.14) which is the desired result. ☐

Proposition 3. *Let* $(r^n_{\mathrm{w},h}, r^n_{\mathrm{nw},h}, r^n_{e,h}, \boldsymbol{f}^n_{u,h}) \in L^{3+d}_h$ *and* $(p^0_{\mathrm{w},h}, p^0_{\mathrm{nw},h}, T^0_h, \boldsymbol{u}^0_h) \in L^{3+d}_h$ *be given. The finite volume scheme (5.2)–(5.4)–(5.14)–(5.8) is equivalent to the variational formulation: Find* $(p^n_{\mathrm{w},h}, p^n_{\mathrm{nw},h}, T^n_h, \boldsymbol{u}^n_h) \in L^{3+d}_h$ *such that for all* $(\varphi_{\mathrm{w},h}, \varphi_{\mathrm{nw},h}, \psi_h, \boldsymbol{\omega}_h) \in H^{3+d}_h$*, we have*

$$
\begin{aligned}
\int_\Omega \rho_{\mathrm{w}} \partial^n_t \left(s_{\mathrm{w},h}\phi_h\right) \varphi_{\mathrm{w},h} \,\mathrm{d}\boldsymbol{x} + \frac{1}{d} \int_\Omega \rho_{\mathrm{w}} k M^n_{\mathrm{w},h} \boldsymbol{\nabla}_h p^n_{\mathrm{w},h} \cdot \boldsymbol{\nabla}_h \varphi_{\mathrm{w},h} \,\mathrm{d}\boldsymbol{x} \\
= \int_\Omega r^n_{\mathrm{w},h} \varphi_{\mathrm{w},h}(\boldsymbol{x}) \,\mathrm{d}\boldsymbol{x}.
\end{aligned}
\tag{5.20}
$$

$$
\begin{aligned}
\int_\Omega \rho_{\mathrm{nw}} \partial^n_t \left(s_{\mathrm{nw},h}\phi_h\right) \varphi_{\mathrm{nw},h} \,\mathrm{d}\boldsymbol{x} + \frac{1}{d} \int_\Omega \rho_{\mathrm{nw}} k M^n_{\mathrm{nw},h} \boldsymbol{\nabla}_h p^n_{\mathrm{nw},h} \cdot \boldsymbol{\nabla}_h \varphi_{\mathrm{nw},h} \,\mathrm{d}\boldsymbol{x} \\
= \int_\Omega r^n_{\mathrm{nw},h} \varphi_{\mathrm{nw},h}(\boldsymbol{x}) \,\mathrm{d}\boldsymbol{x}.
\end{aligned}
\tag{5.21}
$$

$$
\begin{aligned}
&\int_\Omega \partial^n_t (S_{s,h}) \psi_h \,\mathrm{d}\boldsymbol{x} + \int_\Omega \sum_{\alpha\in\{\mathrm{w},\mathrm{nw}\}} \rho_\alpha \frac{s^{n-1}_{\alpha,h}\phi^{n-1}_h}{T^n_h} \partial^n_t (e_{\alpha,h}) \psi_h \,\mathrm{d}\boldsymbol{x} \\
&- \frac{1}{d} \int_\Omega \sum_{\alpha\in\{\mathrm{w},\mathrm{nw}\}} \frac{\rho_\alpha k M^n_{\alpha,h}}{T^n_h} \boldsymbol{\nabla}_h p^n_{\alpha,h} \cdot \boldsymbol{\nabla}_h e^n_{\alpha,h} \psi_h \,\mathrm{d}\boldsymbol{x} \\
&+ \frac{1}{d} \int_\Omega \frac{\lambda}{T_{\mathrm{ref}}} \boldsymbol{\nabla}_h T^n_h \cdot \boldsymbol{\nabla}_h \psi_h \,\mathrm{d}\boldsymbol{x} = \int_\Omega \left(\frac{r^n_{e,h}(\boldsymbol{x})}{T^n_h} - c^n_{\mathrm{w},h} r^n_{\mathrm{w},h}(\boldsymbol{x}) - c^n_{\mathrm{nw},h} r^n_{\mathrm{nw},h} \right) \psi_h(\boldsymbol{x}) \,\mathrm{d}\boldsymbol{x}
\end{aligned}
\tag{5.22}
$$

$$
\begin{aligned}
&\int_\Omega \partial^n_{tt} \boldsymbol{u}_h \cdot \boldsymbol{\omega}_h \,\mathrm{d}\boldsymbol{x} + \frac{1}{d} \int_\Omega \frac{\mathcal{E}}{(1+\nu)} \epsilon(\boldsymbol{u}^n_h) : \epsilon(\boldsymbol{\omega}_h) \,\mathrm{d}\boldsymbol{x} \\
&+ \int_\Omega \frac{\mathcal{E}\nu}{(1+\nu)(1-2\nu)} \mathrm{div}_h\, \boldsymbol{u}^n_h \mathbb{I}_d : \epsilon(\boldsymbol{\omega}_h) \,\mathrm{d}\boldsymbol{x} \\
&- \int_\Omega \left(b\pi^n_h + 3\alpha_s K_s (T^n_h - T_{\mathrm{ref}})\right) \mathrm{div}_h\left(\boldsymbol{\omega}_h\right) \mathrm{d}\boldsymbol{x} = \int_\Omega \boldsymbol{f}^n_{u,h} \cdot \boldsymbol{\omega}_h(\boldsymbol{x}) \,\mathrm{d}\boldsymbol{x}.
\end{aligned}
\tag{5.23}
$$

Proof of Proposition 3. First, we establish (5.20). Let $\varphi_{\mathrm{w},h} \in H_h$ be a test function. We multiply the equation (5.2) by $\varphi_{\mathrm{w},K}$ then we sum over K to obtain

$$
\begin{aligned}
&\sum_{K\in\mathcal{T}} |K| \rho_{\mathrm{w}} (s^n_{\mathrm{w},K}\phi^n_K - s^{n-1}_{\mathrm{w},K}\phi^{n-1}_K) \varphi_{\mathrm{w},K} \\
&+ \delta t \sum_{K\in\mathcal{T}} \sum_{L\in\mathcal{N}(K)} \rho_{\mathrm{w}} k M^n_{\mathrm{w},KL} \tau_{KL} (p^n_{\mathrm{w},K} - p^n_{\mathrm{w},L}) \varphi_{\mathrm{w},K} = \delta t \sum_{K\in\mathcal{T}} |K| r^n_{\mathrm{w},K} \varphi_{\mathrm{w},K}.
\end{aligned}
$$

From the definition of the discrete piecewise constant functions, we have

$$\sum_{K\in\mathcal{T}} |K|\rho_{\mathrm{w}}(s^n_{\mathrm{w},K}\phi^n_K - s^{n-1}_{\mathrm{w},K}\phi^{n-1}_K)\varphi_{\mathrm{w},K} = \int_\Omega \rho_{\mathrm{w}}(s^n_{\mathrm{w},h}\phi^n_h - s^{n-1}_{\mathrm{w},h}\phi^{n-1}_h)\varphi_{\mathrm{w},h}\,\mathrm{d}\boldsymbol{x},$$

and

$$\sum_{K\in\mathcal{T}} |K| r^n_{\mathrm{w},K}\varphi_{\mathrm{w},K} = \int_\Omega r^n_{\mathrm{w},h}\varphi_{\mathrm{w},h}\,\mathrm{d}\boldsymbol{x}.$$

Furthermore, integrating by parts and using the definition of the discrete gradient, we get

$$\begin{aligned}
&\sum_{K\in\mathcal{T}}\sum_{L\in\mathcal{N}(K)} \rho_{\mathrm{w}} k M^n_{\mathrm{w},KL}\tau_{KL}(p^n_{\mathrm{w},K}-p^n_{\mathrm{w},L})\varphi_{\mathrm{w},K}\\
&= d\sum_{\sigma_{KL}\in\mathcal{E}_h} \rho_{\mathrm{w}} k|\mathcal{D}_{KL}|M^n_{\mathrm{w},KL}\frac{p^n_{\mathrm{w},L}-p^n_{\mathrm{w},K}}{d_{KL}}\,\frac{\varphi^n_{\mathrm{w},L}-\varphi^n_{\mathrm{w},K}}{d_{KL}}\\
&= d\sum_{\sigma_{KL}\in\mathcal{E}_h} \rho_{\mathrm{w}} k|\mathcal{D}_{KL}|M^n_{\mathrm{w},KL}\left(\frac{p^n_{\mathrm{w},L}-p^n_{\mathrm{w},K}}{d_{KL}}\boldsymbol{n}_{KL}\right)\cdot\left(\frac{\varphi^n_{\mathrm{w},L}-\varphi^n_{\mathrm{w},K}}{d_{KL}}\right)\boldsymbol{n}_{KL}\\
&= \frac{1}{d}\sum_{\sigma_{KL}\in\mathcal{E}_h} |\mathcal{D}_{KL}|\rho_{\mathrm{w}} k M^n_{\mathrm{w},KL}\boldsymbol{\nabla}_h p^n_{\mathrm{w},h|\mathcal{D}_{KL}}\cdot\boldsymbol{\nabla}_h\varphi_{\mathrm{w},h|\mathcal{D}_{KL}}\\
&= \frac{1}{d}\int_\Omega \rho_{\mathrm{w}} k M^n_{\mathrm{w},h}\boldsymbol{\nabla}_h p^n_{\mathrm{w},h}\cdot\boldsymbol{\nabla}_h\varphi_{\mathrm{w},h}\,\mathrm{d}\boldsymbol{x}
\end{aligned}$$

which establishes (5.20). In the same manner, the variational formulation (5.21) on the non-wetting fluid pressure is obtained by multiplying equation (5.4) by $\varphi_{\mathrm{nw},h}$, summing over K, and integrating by parts. To derive the variational formulation (5.22) on entropy, we use the reformulated discrete entropy conservation equation (5.14). From the definition of the discrete functions, we have

$$\begin{aligned}
&\sum_{K\in\mathcal{T}} |K|\left((S^n_{s,K}-S^{n-1}_{s,K}) + \sum_{\alpha\in\{\mathrm{w},\mathrm{nw}\}}\frac{\rho_\alpha\phi^{n-1}_K s^{n-1}_{\alpha,K}}{T^n_K}(e^n_{\alpha,K}-e^{n-1}_{\alpha,K})\right)\psi_K\\
&= \int_\Omega\left(\partial^n_t(S_{s,h}) + \sum_{\alpha\in\{\mathrm{w},\mathrm{nw}\}}\frac{\rho_\alpha\phi^{n-1}_h s^{n-1}_{\alpha,h}}{T^n_h}\partial^n_t(e_{\alpha,h})\right)\psi_h\,\mathrm{d}\boldsymbol{x},
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{K\in\mathcal{T}} |K|\left(\frac{r^n_{e,K}}{T^n_K} - c^n_{\mathrm{w},K}r^n_{\mathrm{w},K} - c^n_{\mathrm{nw},K}r^n_{\mathrm{nw},K}\right)\psi_K\\
&= \int_\Omega\left(\frac{r^n_{e,h}}{T^n_h} - c^n_{\mathrm{w},h}r^n_{\mathrm{w},h} - c^n_{\mathrm{nw},h}r^n_{\mathrm{nw},h}\right)\psi_h\,\mathrm{d}\boldsymbol{x}.
\end{aligned}$$

Additionally, transforming the sum over cells to the sum over edges and using the definition of the discrete gradient, from the second term of (5.14) we obtain

$$\begin{aligned}
&\frac{1}{2}\sum_{K\in\mathcal{T}}\sum_{L\in\mathcal{N}(K)}\sum_{\alpha\in\{\mathrm{w},\mathrm{nw}\}}\frac{\rho_\alpha k M^n_{\alpha,KL}\tau_{KL}}{T^n_{KL}}(e^n_{\alpha,L}-e^n_{\alpha,K})(p^n_{\alpha,K}-p^n_{\alpha,L})\psi_{KL}\\
&= d\sum_{\sigma_{KL}\in\mathcal{E}_h}\sum_{\alpha\in\{\mathrm{w},\mathrm{nw}\}}\frac{\rho_\alpha k M^n_{\alpha,KL}}{T^n_{KL}}|\mathcal{D}_{KL}|\frac{e^n_{\alpha,L}-e^n_{\alpha,K}}{d_{KL}}\,\frac{p^n_{\alpha,K}-p^n_{\alpha,L}}{d_{KL}}\psi_{KL}\\
&= -\frac{1}{d}\sum_{\sigma_{KL}\in\mathcal{E}_h}|\mathcal{D}_{KL}|\sum_{\alpha\in\{\mathrm{w},\mathrm{nw}\}}\frac{\rho_\alpha k M^n_{\alpha,KL}}{T^n_{KL}}\boldsymbol{\nabla}_h e^n_{\alpha,h|\mathcal{D}_{KL}}\cdot\boldsymbol{\nabla}_h p^n_{\alpha,h|\mathcal{D}_{KL}}\psi_{KL}\\
&= -\frac{1}{d}\int_\Omega\sum_{\alpha\in\{\mathrm{w},\mathrm{nw}\}}\frac{\rho_\alpha k M^n_{\alpha,h}}{T^n_h}\boldsymbol{\nabla}_h e^n_{\alpha,h}\cdot\boldsymbol{\nabla}_h p^n_{\alpha,h}\psi_h\,\mathrm{d}\boldsymbol{x}.
\end{aligned}$$

Furthermore, integrating by parts and using the definition of the discrete gradient, we get

$$\sum_{K\in\mathcal{T}}\sum_{L\in\mathcal{N}(K)}\frac{\lambda}{T_{\mathrm{ref}}}\tau_{KL}(T_K^n-T_L^n)\psi_K=\frac{1}{d}\int_\Omega\frac{\lambda}{T_{\mathrm{ref}}}\boldsymbol{\nabla}_hT_h^n\cdot\boldsymbol{\nabla}_h\psi_h\,\mathrm{d}\boldsymbol{x}.$$

Finally, let $\boldsymbol{\omega}_h\in H_h^d$ be a test function. To obtain equation (5.23), we multiply equation (5.8) by $\boldsymbol{\omega}_K$ then by summing over K and following the same arguments as in [35, Proposition 4] by replacing $(p_K^n)_K$ by $(\pi_K^n)_K$, we conclude the proof. ☐

6 Discrete energy estimates

In this section, we derive the discrete energy estimates for the model under the assumptions introduced in Section 2.2 on the discrete physical data. We begin by establishing, in the following lemma, that the saturations remain positive at the discrete level.

Lemma 3. *Let* $\left(s_{\alpha,K}^0\right)_{K\in\mathcal{T}}\in[0,1]$ *for* $\alpha\in\{\mathrm{w},\mathrm{nw}\}$*. Assume that* $(p_{\mathrm{w},\mathscr{D}},p_{\mathrm{nw},\mathscr{D}},T_{\mathscr{D}},\boldsymbol{u}_{\mathscr{D}})$ *is a solution of the finite volume scheme* (5.2)-(5.8) *and let the assumptions of Section 2.2 hold. Then, the saturation* $s_{\alpha,K}^n$ *remains in the interval* $[0,1]$ *for all* $K\in\mathcal{T}$ *and* $n\in\{1,\dots,N_T\}$*.*

Proof of Lemma 3. We prove by induction on n that $s_{\alpha,K}^n\geq 0$ for all $K\in\mathcal{T}$. Let us start with $\alpha=\mathrm{w}$. For $n=0$, the property follows directly from the hypothesis of the lemma, since $\left(s_{\mathrm{w},K}^0\right)_{K\in\mathcal{T}}\in[0,1]$. Let $n\in[1,N_T]$ and assume that $s_{\mathrm{w},K}^{n-1}\geq 0$. We aim to prove that $s_{\mathrm{w},K}^n\geq 0$. To this end, we consider the control volume K^* such that $s_{\mathrm{w},K^*}^n=\min\{s_{\mathrm{w},L}^n\}_{L\in\mathcal{T}}$ and we multiply the discrete wetting fluid mass conservation equation (5.2) by $(s_{\mathrm{w},K^*}^n)^-=\max(-s_{\mathrm{w},K^*}^n,0)$ to obtain

$$\begin{aligned}&|K^*|\rho_\mathrm{w}\frac{\phi_{K^*}^ns_{\mathrm{w},K^*}^n-\phi_{K^*}^{n-1}s_{\mathrm{w},K^*}^{n-1}}{\delta t}(s_{\mathrm{w},K^*}^n)^-\\&+\sum_{L\in\mathcal{N}(K^*)}\rho_\mathrm{w}k\tau_{K^*L}M_{\mathrm{w},K^*L}^n(p_{\mathrm{w},K^*}^n-p_{\mathrm{w},L}^n)(s_{\mathrm{w},K^*}^n)^-=r_{\mathrm{w},K^*}^n(s_{\mathrm{w},K^*}^n)^-.\end{aligned}\tag{6.1}$$

Expanding the first term of equation (6.1) and using that $s_{\mathrm{w},K^*}^{n-1}\geq 0$ together with Assumption (**A1**), we get

$$\begin{aligned}&\frac{|K^*|\rho_\mathrm{w}}{\delta t}\left(\phi_{K^*}^ns_{\mathrm{w},K^*}^n-\phi_{K^*}^{n-1}s_{\mathrm{w},K^*}^{n-1}\right)(s_{\mathrm{w},K^*}^n)^-\\&=-\frac{|K^*|\rho_\mathrm{w}}{\delta t}\left(\phi_{K^*}^n\left|(s_{\mathrm{w},K^*}^n)^-\right|^2+\phi_{K^*}^{n-1}s_{\mathrm{w},K^*}^{n-1}(s_{\mathrm{w},K^*}^n)^-\right)\leq 0.\end{aligned}\tag{6.2}$$

Now, to determine the sign of the second term of (6.1). We distinguish two cases based on the sign of $p_{\mathrm{w},K^*}^n-p_{\mathrm{w},L}^n$.

- **Case 1:** $p_{\mathrm{w},K^*}^n-p_{\mathrm{w},L}^n\geq 0$**.** In this case, using the definition of the mobility on the interfaces given by (5.3) and Assumption (**A6**), we obtain that

$$M_\mathrm{w}(s_{\mathrm{w},K^*L}^n)(s_{\mathrm{w},K^*}^n)^-=M_\mathrm{w}(s_{\mathrm{w},K^*}^n)(s_{\mathrm{w},K^*}^n)^-=0.$$

- **Case 2:** $p_{\mathrm{w},K^*}^n-p_{\mathrm{w},L}^n<0$**.** In this case, we obtain from the definition of the mobility on the interface given by (5.3), the choice of the control volume K^*, and from Assumption (**A6**) that

$$M_\mathrm{w}(s_{\mathrm{w},K^*L}^n)=M_\mathrm{w}(s_{\mathrm{w},L}^n)\geq M_\mathrm{w}(s_{\mathrm{w},K^*}^n).$$

  Then, from the second term of (6.1), we obtain

$$\begin{aligned}&\rho_\mathrm{w}k\tau_{K^*L}(p_{\mathrm{w},K^*}^n-p_{\mathrm{w},L}^n)M_\mathrm{w}(s_{\mathrm{w},K^*L}^n)(s_{\mathrm{w},K^*}^n)^-\\&\leq\rho_\mathrm{w}k\tau_{K^*L}(p_{\mathrm{w},K^*}^n-p_{\mathrm{w},L}^n)M_\mathrm{w}(s_{\mathrm{w},K^*}^n)(s_{\mathrm{w},K^*}^n)^-=0.\end{aligned}$$

In both cases, we get that the second term of equation (6.1) is non-positive. Furthermore, from Assumption (**A3**) on the source terms, we obtain that $r^n_{\mathrm{w},K^*}(s^n_{\mathrm{w},K^*})^- \geq 0$. Consequently, we obtain from (6.1) that

$$|K^*|\rho_{\mathrm{w}} \frac{\phi^n_{K^*} s^n_{\mathrm{w},K^*} - \phi^{n-1}_{K^*} s^{n-1}_{\mathrm{w},K^*}}{\delta t} (s^n_{\mathrm{w},K^*})^- \geq 0.$$

And since we have from (6.2) that it is non-positive, then we obtain that $(s^n_{\mathrm{w},K^*})^- = 0$. Thus, we deduce that $s^n_{\mathrm{w},K} \geq 0$, for all $K \in \mathcal{T}$. Similarly, we can show that $s^n_{\mathrm{nw},K} \geq 0$ using the non-wetting fluid discrete mass conservation equation (5.4). Moreover, from the relation $s^n_{\mathrm{w},K} + s^n_{\mathrm{nw},K} = 1$, we obtain that $s^n_{\alpha,K} \in [0,1]$ for $\alpha \in \{\mathrm{w}, \mathrm{nw}\}$, which concludes the proof. □

Now, the following Lemma provides a key estimate that will be used to derive the discrete energy estimates.

Lemma 4. *Let* $(\boldsymbol{u}^n_h, p^n_{\mathrm{w},h}, p^n_{\mathrm{nw},h}, T^n_h)$ *be the solution to the system of equations* (5.20)–(5.23) *and assume that the assumption in Section 2.2 hold. We have the following inequality:*

$$\begin{aligned}
&\int_\Omega \partial^n_t E_h \,\mathrm{d}\boldsymbol{x} + \int_\Omega \frac{m_0}{2} \partial^n_t \left(|\partial^n_t \boldsymbol{u}_h|^2\right) \mathrm{d}\boldsymbol{x} + \frac{1}{d}\int_\Omega \frac{\lambda}{T_{\mathrm{ref}}} |\boldsymbol{\nabla}_h T^n_h|^2 \,\mathrm{d}\boldsymbol{x} \\
&+ \frac{1}{d}\int_\Omega k\left(M^n_{\mathrm{w},h}|\boldsymbol{\nabla}_h p^n_{\mathrm{w},h}|^2 + M^n_{\mathrm{nw},h}|\boldsymbol{\nabla}_h p^n_{\mathrm{nw},h}|^2\right) \mathrm{d}\boldsymbol{x} \\
&\leq \int_\Omega \left(\sum_{\alpha\in\{\mathrm{w},\mathrm{nw}\}} r^n_{\alpha,h}\left(\frac{p^n_{\alpha,h}}{\rho_\alpha} + e^n_{\alpha,h} - c^n_{\alpha,h} T^n_h\right) + r^n_{e,h} + \boldsymbol{f}^n_{u,h}\cdot\partial^n_t \boldsymbol{u}_h \right) \mathrm{d}\boldsymbol{x},
\end{aligned} \tag{6.3}$$

where E^n_h *is the total energy of the system and it is defined by*

$$\begin{aligned}
E^n_h &:= \phi^n_h U^n_h + \frac{1}{2}\begin{bmatrix}\pi^n_h & T^n_h\end{bmatrix}\mathbb{M}\begin{bmatrix}\pi^n_h \\ T^n_h\end{bmatrix} + \frac{\mathcal{E}}{2(1+\nu)}\left[|\epsilon(\boldsymbol{u}^n_h)|^2 + \frac{\nu}{1-2\nu}(\operatorname{div}\boldsymbol{u}^n_h)^2\right] \\
&+ \sum_{\alpha\in\{\mathrm{w},\mathrm{nw}\}} \phi^n_h s^n_{\alpha,h}\rho_\alpha e^n_{\alpha,h},
\end{aligned}$$

where $\mathbb{M}$ *is defined by* (3.4).

Proof of Lemma 4. First, we consider in (5.20)–(5.23), the test functions $\varphi_{\mathrm{w},h} = \left(\frac{p^n_{\mathrm{w},h}}{\rho_{\mathrm{w}}} + e^n_{\mathrm{w},h}\right) \in H_h$, $\varphi_{\mathrm{nw},h} = \left(\frac{p^n_{\mathrm{nw},h}}{\rho_{\mathrm{nw}}} + e^n_{\mathrm{nw},h}\right) \in H_h$, $\psi_h = T^n_h \in H_h$ and $\boldsymbol{\omega}_h = \partial^n_t \boldsymbol{u}_h \in H^d_h$. Then, we sum these equations to obtain

$$E_1 + E_2 + E_3 + E_4 + E_5 + E_6 + E_7 + E_8 = E_9, \tag{6.4}$$

where

$$
\begin{aligned}
E_1 &:= \int_\Omega \partial_t^n \left(s_{\mathrm{w},h}\phi_h\right)\left(p_{\mathrm{w},h}^n + \rho_{\mathrm{w}} e_{\mathrm{w},h}^n\right)\,\mathrm{d}\boldsymbol{x} + \int_\Omega \rho_{\mathrm{w}} s_{\mathrm{w},h}^{n-1}\phi_h^{n-1}\partial_t^n(e_{\alpha,h})\,\mathrm{d}\boldsymbol{x},\\
E_2 &:= \int_\Omega \partial_t^n \left(s_{\mathrm{nw},h}\phi_h\right)\left(p_{\mathrm{nw},h}^n + \rho_{\mathrm{nw}} e_{\mathrm{nw},h}^n\right)\,\mathrm{d}\boldsymbol{x} + \int_\Omega \rho_{\mathrm{nw}} s_{\mathrm{nw},h}^{n-1}\phi_h^{n-1}\partial_t^n(e_{\alpha,h})\,\mathrm{d}\boldsymbol{x},\\
E_3 &:= \int_\Omega \partial_t^n(S_{s,h})T_h^n\,\mathrm{d}\boldsymbol{x},\\
E_4 &:= \frac{1}{d}\int_\Omega \rho_{\mathrm{w}} k M_{\mathrm{w},h}^n \boldsymbol{\nabla}_h p_{\mathrm{w},h}^n \cdot \boldsymbol{\nabla}_h\left(\frac{p_{\mathrm{w},h}^n}{\rho_{\mathrm{w}}} + e_{\mathrm{w},h}^n\right)\,\mathrm{d}\boldsymbol{x} - \frac{1}{d}\int_\Omega \rho_{\mathrm{w}} k M_{\mathrm{w},h}^n \boldsymbol{\nabla}_h p_{\mathrm{w},h}^n \cdot \boldsymbol{\nabla}_h e_{\mathrm{w},h}^n\,\mathrm{d}\boldsymbol{x},\\
E_5 &:= \frac{1}{d}\int_\Omega \rho_{\mathrm{nw}} k M_{\mathrm{nw},h}^n \boldsymbol{\nabla}_h p_{\mathrm{nw},h}^n \cdot \boldsymbol{\nabla}_h\left(\frac{p_{\mathrm{nw},h}^n}{\rho_{\mathrm{nw}}} + e_{\mathrm{nw},h}^n\right)\,\mathrm{d}\boldsymbol{x} - \frac{1}{d}\int_\Omega \rho_{\mathrm{nw}} k M_{\mathrm{nw},h}^n \boldsymbol{\nabla}_h p_{\mathrm{nw},h}^n \cdot \boldsymbol{\nabla}_h e_{\mathrm{nw},h}^n\,\mathrm{d}\boldsymbol{x},\\
E_6 &:= \frac{1}{d}\int_\Omega \frac{\lambda}{T_{\mathrm{ref}}}\boldsymbol{\nabla}_h T_h^n \cdot \boldsymbol{\nabla}_h T_h^n\,\mathrm{d}\boldsymbol{x},\\
E_7 &:= \int_\Omega m_0 \partial_{tt}^n \boldsymbol{u}_h \cdot \partial_t^n \boldsymbol{u}_h\,\mathrm{d}\boldsymbol{x} + \frac{1}{d}\int_\Omega \frac{\mathcal{E}}{(1+\nu)}\epsilon(\boldsymbol{u}_h^n) : \epsilon(\partial_t^n \boldsymbol{u}_h)\,\mathrm{d}\boldsymbol{x} + \int_\Omega \frac{\mathcal{E}\nu}{(1+\nu)(1-2\nu)}\mathrm{div}_h\, \boldsymbol{u}_h^n \mathbb{I}_d : \epsilon(\partial_t^n \boldsymbol{u}_h)\,\mathrm{d}\boldsymbol{x},\\
E_8 &:= -\int_\Omega \left(b\pi_h^n + 3\alpha_s K_s(T_h^n - T_{\mathrm{ref}})\right)\mathrm{div}_h\left(\partial_t^n \boldsymbol{u}_h\right)\mathrm{d}\boldsymbol{x},\\
E_9 &:= \int_\Omega r_{\mathrm{w},h}^n\left(\frac{p_{\mathrm{w},h}^n}{\rho_{\mathrm{w}}} + e_{\mathrm{w},h}^n - c_{\mathrm{w},h}^n T_h^n\right) + r_{\mathrm{nw},h}^n\left(\frac{p_{\mathrm{nw},h}^n}{\rho_{\mathrm{nw}}} + e_{\mathrm{nw},h}^n - c_{\mathrm{nw},h}^n T_h^n + r_{e,h}^n + \boldsymbol{f}_{u,h}^n \cdot \partial_t^n \boldsymbol{u}_h\right)\mathrm{d}\boldsymbol{x}.
\end{aligned}
$$

Expanding E_1, we obtain

$$
\begin{aligned}
E_1 &= \int_\Omega \left(s_{\mathrm{w},h}^n p_{\mathrm{w},h}^n \partial_t^n \phi_h + \phi_h^{n-1} p_{\mathrm{w},h}^n \partial_t^n s_{\mathrm{w},h}\right)\mathrm{d}\boldsymbol{x} + \int_\Omega \rho_{\mathrm{w}} \partial_t^n(s_{\mathrm{w},h}\phi_h e_{\mathrm{w},h})\,\mathrm{d}\boldsymbol{x}\\
&:= E_{1,1} + E_{1,2}.
\end{aligned}
$$

In the same manner, expanding E_2, we obtain

$$
\begin{aligned}
E_2 &= \int_\Omega \left(s_{\mathrm{nw},h}^n p_{\mathrm{nw},h}^n \partial_t^n \phi_h + \phi_h^{n-1} p_{\mathrm{nw},h}^n \partial_t^n s_{\mathrm{nw},h}\right)\mathrm{d}\boldsymbol{x} + \int_\Omega \rho_{\mathrm{nw}} \partial_t^n(s_{\mathrm{nw},h}\phi_h e_{\mathrm{nw},h})\,\mathrm{d}\boldsymbol{x}\\
&:= E_{2,1} + E_{2,2}.
\end{aligned}
$$

Furthermore, summing $E_{1,1}$ and $E_{2,1}$ and employing the relation (2.3) together with the definition of capillary pressure (2.4) and the discrete averaged fluid pressure (5.13), we obtain

$$
\begin{aligned}
E_{1,1} + E_{2,1} &= \int_\Omega \left(p_h^{\star,n}\partial_t^n \phi_h - \phi_h^{n-1} p_{\mathrm{c},h}^n \partial_t^n s_{\mathrm{w},h}\right)\mathrm{d}\boldsymbol{x}\\
&= \int_\Omega \left(p_h^{\star,n}\partial_t^n \phi_h - \phi_h^{n-1}\partial_t^n(p_{\mathrm{c},h}s_{\mathrm{w},h}) + \phi_h^{n-1} s_{\mathrm{w},h}^{n-1}\partial_t^n p_{\mathrm{c},h}\right)\mathrm{d}\boldsymbol{x}.
\end{aligned}
\tag{6.5}
$$

In addition, from the mean value theorem, we have

$$
p_{\mathrm{c},h}^n - p_{\mathrm{c},h}^{n-1} = p_{\mathrm{c},h}'(z_h)(s_h^n - s_h^{n-1}), \tag{6.6}
$$

with $z_h \in]s_h^{n-1}, s_h^n[$ if $s_{\mathrm{w},h}^n \geq s_{\mathrm{w},h}^{n-1}$ and $z_h \in]s_h^n, s_h^{n-1}[$ otherwise.
We also have

$$
\mathcal{N}_h^n - \mathcal{N}_h^{n-1} = \mathcal{N}_h'(z_h)(s_h^n - s_h^{n-1}), \tag{6.7}
$$

where $\mathcal{N}_h^n = \mathcal{N}(s_h^n)$ and it is defined by (3.1). Since $\mathcal{N}$ is the primitive of $s_{\mathrm{w}} p_{\mathrm{c}}'(s_{\mathrm{w}})$ over $[s_{\mathrm{w}}, 1]$, we find that $\mathcal{N}_h'(z_h) = -z_h p_{\mathrm{c},h}'(z_h)$. Moreover, we distinguish two cases depending on the monotonicity of the wetting phase saturation $s_{\mathrm{w},h}$: **Case 1:** $s_{\mathrm{w},h}^n \geq s_{\mathrm{w},h}^{n-1}$ and **Case 2:** $s_{\mathrm{w},h}^n < s_{\mathrm{w},h}^{n-1}$.

- **Case 1:** $s_{\mathrm{w},h}^n \geq s_{\mathrm{w},h}^{n-1}$. Since $z_h \in]s_h^{n-1}, s_h^n[$ and the capillary pressure is a decreasing function, we obtain that

  $z_h p_{\mathrm{c},h}'(z_h)(s_{\mathrm{w},h}^n - s_{\mathrm{w},h}^{n-1}) \leq s_{\mathrm{w},h}^{n-1} p_{\mathrm{c},h}'(z_h)(s_{\mathrm{w},h}^n - s_{\mathrm{w},h}^{n-1}).$

- **Case 2:** $s^n_{\mathrm{w},h} < s^{n-1}_{\mathrm{w},h}$. We have that $z_h \in]s^n_h, s^{n-1}_h[$ and that the capillary pressure is a decreasing function then we obtain that $z_h p'_{\mathrm{c},h}(z_h)(s^n_{\mathrm{w},h} - s^{n-1}_{\mathrm{w},h}) \leq s^{n-1}_{\mathrm{w},h} p'_{\mathrm{c},h}(z_h)(s^n_{\mathrm{w},h} - s^{n-1}_{\mathrm{w},h})$.

In both cases, we find that

$$z_h p'_{\mathrm{c},h}(z_h)(s^n_{\mathrm{w},h} - s^{n-1}_{\mathrm{w},h}) \leq s^{n-1}_{\mathrm{w},h} p'_{\mathrm{c},h}(z_h)(s^n_{\mathrm{w},h} - s^{n-1}_{\mathrm{w},h}). \tag{6.8}$$

Furthermore, combining (6.5), (6.6), (6.7) and (6.8) together with the fact that $\mathcal{N}'_h(z_h) = -z_h p'_{\mathrm{c},h}(z_h)$, we obtain

$$\begin{aligned}
E_{1,1} + E_{2,1} &= \int_\Omega \left(p^{\star,n}_h \partial^n_t \phi_h - \phi^{n-1}_h \partial^n_t (p_{\mathrm{c},h} s_{\mathrm{w},h}) + \phi^{n-1}_h s^{n-1}_{\mathrm{w},h} p'_{\mathrm{c},h}(z_h) \partial^n_t s_{\mathrm{w},h} \right) \mathrm{d}\boldsymbol{x} \\
&\geq \int_\Omega \left(p^{\star,n}_h \partial^n_t \phi_h - \phi^{n-1}_h \partial^n_t (p_{\mathrm{c},h} s_{\mathrm{w},h}) + \phi^{n-1}_h z_h p'_{\mathrm{c},h}(z_h) \partial^n_t s_{\mathrm{w},h} \right) \mathrm{d}\boldsymbol{x} \\
&= \int_\Omega \left(p^{\star,n}_h \partial^n_t \phi_h - \phi^{n-1}_h \partial^n_t (p_{\mathrm{c},h} s_{\mathrm{w},h}) - \phi^{n-1}_h \mathcal{N}'_h(z_h) \partial^n_t s_{\mathrm{w},h} \right) \mathrm{d}\boldsymbol{x} \\
&= \int_\Omega \left(p^{\star,n}_h \partial^n_t \phi_h - \phi^{n-1}_h \partial^n_t (p_{\mathrm{c},h} s_{\mathrm{w},h}) - \phi^{n-1}_h \partial^n_t \mathcal{N}_h \right) \mathrm{d}\boldsymbol{x}.
\end{aligned} \tag{6.9}$$

Employing, in equation (6.9), the fact that $f(b) - f(a) = \int_a^b f'(x)\,\mathrm{d}x$ for all $a, b \in \mathbb{R}$ and $f \in C^1([a,b])$, together with the definition of the function $\mathcal{N}$ in (3.1) and the interfacial energy U^n_h in (5.13), we obtain

$$\begin{aligned}
E_{1,1} + E_{2,1} &\geq \int_\Omega \left(p^{\star,n}_h \partial^n_t \phi_h - \phi^{n-1}_h \int_{s^{n-1}_{\mathrm{w},h}}^{s^n_{\mathrm{w},h}} (z p_{\mathrm{c}}(z))'\,\mathrm{d}z + \phi^{n-1}_h \int_{s^{n-1}_{\mathrm{w},h}}^{s^n_{\mathrm{w},h}} z p'_{\mathrm{c}}(z)\,\mathrm{d}z \right) \mathrm{d}\boldsymbol{x} \\
&= \int_\Omega \left(p^{\star,n}_h \partial^n_t \phi_h - \phi^{n-1}_h \int_{s^{n-1}_{\mathrm{w},h}}^{s^n_{\mathrm{w},h}} p_{\mathrm{c}}(z)\,\mathrm{d}z \right) \mathrm{d}\boldsymbol{x} \\
&= \int_\Omega \left(p^{\star,n}_h \partial^n_t \phi_h + \phi^{n-1}_h \partial^n_t U_h \right) \mathrm{d}\boldsymbol{x} \\
&= \int_\Omega \left(\pi^n_h \partial^n_t \phi_h + \partial^n_t (\phi_h U_h) \right) \mathrm{d}\boldsymbol{x}.
\end{aligned}$$

Consequently, we obtain

$$E_1 + E_2 \geq \int_\Omega \left(\pi^n_h \partial^n_t \phi_h + \partial^n_t (\phi_h U_h) \right) \mathrm{d}\boldsymbol{x} + \sum_{\alpha \in \{\mathrm{w},\mathrm{nw}\}} \int_\Omega \rho_\alpha \partial^n_t (s_{\alpha,h} \phi_h e_{\alpha,h})\,\mathrm{d}\boldsymbol{x}. \tag{6.10}$$

Furthermore, employing the definition of the discrete porosity and the skeleton entropy (5.9), we obtain

$$\begin{aligned}
&E_3 + E_8 + \int_\Omega \pi^n_h \partial^n_t \phi_h \,\mathrm{d}\boldsymbol{x} \\
&= \int_\Omega \left(\pi^n_h \partial^n_t \phi_h + T^n_h \partial^n_t S_{s,h} - (b \pi^n_h + 3\alpha_s K_s (T^n_h - T_{\mathrm{ref}}))\, \mathrm{div}_h\, (\partial^n_t \boldsymbol{u}_h) \right) \mathrm{d}\boldsymbol{x} \\
&= \sum_{K \in \mathcal{T}} |K| \left(\pi^n_K \partial^n_t \phi_K + T^n_K \partial^n_t S_{s,K} - (b \pi^n_K + 3\alpha_s K_s (T^n_K - T_{\mathrm{ref}}))\, \mathrm{div}_K\, (\partial^n_t \boldsymbol{u}_h) \right) \\
&= \sum_{K \in \mathcal{T}} |K| \left(-3\alpha_\phi \pi^n_K \partial^n_t T_K + \frac{1}{N} \pi^n_K \partial^n_t \pi_K - 3\alpha_\phi T^n_K \partial^n_t \pi_K \right) \\
&+ \sum_{K \in \mathcal{T}} |K| \left(\frac{C_s}{T_{\mathrm{ref}}} T^n_K \partial^n_t T_K + 3\alpha_s K_s T_{\mathrm{ref}} \mathrm{div}_K\, (\partial^n_t (\boldsymbol{u}_h)) \right).
\end{aligned} \tag{6.11}$$

In addition, from the definition of the discrete divergence, we have that $\sum_{K \in \mathcal{T}} |K| \mathrm{div}_K\, \partial^n_t \boldsymbol{u}_h =$

$\int_\Omega \operatorname{div}_h \partial_t^n \boldsymbol{u}_h = 0$ since $\boldsymbol{u}_h^n = 0$ on $\partial\Omega$ for all $n \in [0, N_T]$. Consequently, equation (6.11) reads

$$\begin{aligned}
&E_3 + E_8 + \int_\Omega \pi_h^n \partial_t^n \phi_h \,\mathrm{d}\boldsymbol{x} \\
&= \int_\Omega \left(-3\alpha_\phi \pi_h^n \partial_t^n T_h + \frac{1}{N} \pi_h^n \partial_t^n \pi_h - 3\alpha_\phi T_h^n \partial_t^n \pi_h + \frac{C_s}{T_{\text{ref}}} T_h^n \partial_t^n T_h \right) \mathrm{d}\boldsymbol{x} \\
&= \int_\Omega \frac{1}{\delta t} \left[\begin{bmatrix} \pi_h^n & T_h^n \end{bmatrix} \mathbb{M} \begin{bmatrix} \pi_h^n - \pi_h^{n-1} \\ T_h^n - T_h^{n-1} \end{bmatrix} \right] \mathrm{d}\boldsymbol{x}.
\end{aligned}$$

Moreover, since $\mathbb{M}$ is a real, symmetric, positive definite matrix, we have the following inequality

$$\begin{aligned}
&\frac{1}{\delta t} \left[\begin{bmatrix} \pi_h^n & T_h^n \end{bmatrix} \mathbb{M} \begin{bmatrix} \pi_h^n - \pi_h^{n-1} \\ T_h^n - T_h^{n-1} \end{bmatrix} \right] \\
&\geq \frac{1}{2\delta t} \left[\begin{bmatrix} \pi_h^n & T_h^n \end{bmatrix} \mathbb{M} \begin{bmatrix} \pi_h^n \\ T_h^n \end{bmatrix} - \begin{bmatrix} \pi_h^{n-1} & T_h^{n-1} \end{bmatrix} \mathbb{M} \begin{bmatrix} \pi_h^{n-1} \\ T_h^{n-1} \end{bmatrix} \right] \\
&= \frac{1}{2} \partial_t^n \left[\begin{bmatrix} \pi_h & T_h \end{bmatrix} \mathbb{M} \begin{bmatrix} \pi_h \\ T_h \end{bmatrix} \right].
\end{aligned}$$

This yields

$$E_3 + E_8 + \int_\Omega \pi_h^n \partial_t^n \phi_h \,\mathrm{d}\boldsymbol{x} \geq \frac{1}{2} \partial_t^n \left[\begin{bmatrix} \pi_h & T_h \end{bmatrix} \mathbb{M} \begin{bmatrix} \pi_h \\ T_h \end{bmatrix} \right]. \tag{6.12}$$

Furthermore, we recall the inequality $a(a-b) \geq \frac{1}{2}(a^2 - b^2)$ for all $a, b \in \mathbb{R}$, which yields the bound

$$\partial_{tt}^n \boldsymbol{u}_h \cdot \partial_t^n \boldsymbol{u}_h = \frac{1}{\delta t} \left(\partial_t^n \boldsymbol{u}_h - \partial_t^{n-1} \boldsymbol{u}_h \right) \cdot \partial_t^n \boldsymbol{u}_h \geq \frac{1}{2} \partial_t^n |\partial_t^n \boldsymbol{u}_h|^2. \tag{6.13}$$

In the same way, the second term of E_7 satisfies

$$\epsilon(\boldsymbol{u}_h^n) : \epsilon(\partial_t^n \boldsymbol{u}_h) = \frac{1}{\delta t} \epsilon(\boldsymbol{u}_h^n) : \epsilon(\boldsymbol{u}_h^n - \boldsymbol{u}_h^{n-1}) \geq \frac{1}{2} \partial_t^n (|\epsilon(\boldsymbol{u}_h)|^2). \tag{6.14}$$

Likewise, the third term of E_7 satisfies

$$\operatorname{div}_h \boldsymbol{u}_h^n \mathbb{I}_d : \epsilon(\partial_t \boldsymbol{u}_h^n) = \frac{1}{\delta t} \left(\operatorname{div}_h \boldsymbol{u}_h^n \mathbb{I}_d : \epsilon(\boldsymbol{u}_h^n - \boldsymbol{u}_h^{n-1}) \right) \geq \frac{1}{2} \partial_t^n (\operatorname{div}_h \boldsymbol{u}_h)^2. \tag{6.15}$$

Combining the three estimates (6.13), (6.14) and (6.15) yields

$$\begin{aligned}
E_7 \geq{}& \frac{1}{2} \int_\Omega m_0 \partial_t^n |\partial_t^n \boldsymbol{u}_h|^2 \,\mathrm{d}\boldsymbol{x} \\
&+ \frac{1}{2} \int_\Omega \left[\frac{\mathcal{E}}{d(1+\nu)} \partial_t^n (|\epsilon(\boldsymbol{u}_h)|^2) + \frac{\mathcal{E}\nu}{(1+\nu)(1-2\nu)} \partial_t^n (\operatorname{div}_h \boldsymbol{u}_h)^2 \right] \mathrm{d}\boldsymbol{x}.
\end{aligned} \tag{6.16}$$

Now, to obtain the estimate on the gradient terms, we expand E_4 to obtain

$$E_4 = \frac{1}{d} \int_\Omega k M_{\mathrm{w},h}^n \boldsymbol{\nabla}_h p_{\mathrm{w},h}^n \cdot \boldsymbol{\nabla}_h p_{\mathrm{w},h}^n \,\mathrm{d}\boldsymbol{x} = \frac{1}{d} \int_\Omega k M_{\mathrm{w},h}^n |\boldsymbol{\nabla}_h p_{\mathrm{w},h}^n|^2 \,\mathrm{d}\boldsymbol{x}. \tag{6.17}$$

In the same manner, we obtain

$$E_5 = \frac{1}{d} \int_\Omega k M_{\mathrm{nw},h}^n \boldsymbol{\nabla}_h p_{\mathrm{nw},h}^n \cdot \boldsymbol{\nabla}_h p_{\mathrm{nw},h}^n \,\mathrm{d}\boldsymbol{x} = \frac{1}{d} \int_\Omega k M_{\mathrm{nw},h}^n |\boldsymbol{\nabla}_h p_{\mathrm{nw},h}^n|^2 \,\mathrm{d}\boldsymbol{x}. \tag{6.18}$$

Finally, combining equations (6.10), (6.12), (6.16), (6.17) and (6.18) with equation (6.4), we obtain

$$
\begin{aligned}
&\int_\Omega \left(\partial_t^n(\phi_h U_h) + \sum_{\alpha\in\{\mathrm{w},\mathrm{nw}\}} \rho_\alpha \partial_t^n (s_{\alpha,h}\phi_h e_{\alpha,h}) + \frac{1}{2}\partial_t^n \left[\begin{bmatrix}\pi_h & T_h\end{bmatrix} \mathbb{M} \begin{bmatrix}\pi_h \\ T_h\end{bmatrix} \right] \right) \mathrm{d}\boldsymbol{x} \\
&+ \frac{1}{2}\int_\Omega \left[m_0 \partial_t^n |\partial_t^n \boldsymbol{u}_h|^2 + \frac{\mathcal{E}}{d(1+\nu)} \partial_t^n (|\epsilon(\boldsymbol{u}_h)|^2) + \frac{\mathcal{E}\nu}{(1+\nu)(1-2\nu)} \partial_t^n (\operatorname{div}_h \boldsymbol{u}_h)^2 \right] \mathrm{d}\boldsymbol{x} \\
&+ \frac{1}{d}\int_\Omega k M_{\mathrm{w},h}^n |\boldsymbol{\nabla}_h p_{\mathrm{w},h}^n|^2 \,\mathrm{d}\boldsymbol{x} + \frac{1}{d}\int_\Omega k M_{\mathrm{nw},h}^n |\boldsymbol{\nabla}_h p_{\mathrm{nw},h}^n|^2 \,\mathrm{d}\boldsymbol{x} + \frac{1}{d}\int_\Omega \frac{\lambda}{T_{\mathrm{ref}}} |\boldsymbol{\nabla}_h T_h^n|^2 \,\mathrm{d}\boldsymbol{x} \\
&\le \int_\Omega \left(\sum_{\alpha\in\{\mathrm{w},\mathrm{nw}\}} r_{\alpha,h}^n \left(\frac{p_{\alpha,h}^n}{\rho_\alpha} + e_{\alpha,h}^n - c_{\alpha,h}^n T_h^n \right) + r_{e,h}^n + \boldsymbol{f}_{u,h}^n \cdot \partial_t^n \boldsymbol{u}_h \right) \mathrm{d}\boldsymbol{x}.
\end{aligned}
$$

Setting $E_h^n := \phi_h^n U_h^n + \frac{1}{2}\begin{bmatrix}\pi_h^n & T_h^n\end{bmatrix} \mathbb{M} \begin{bmatrix}\pi_h^n \\ T_h^n\end{bmatrix} + \frac{\mathcal{E}}{2(1+\nu)}\left[|\epsilon(\boldsymbol{u}_h^n)|^2 + \frac{\nu}{1-2\nu}(\operatorname{div}\boldsymbol{u}_h^n)^2 \right] + \sum_{\alpha\in\{\mathrm{w},\mathrm{nw}\}} \phi_h^n s_{\alpha,h}^n \rho_\alpha e_{\alpha,h}^n$, it concludes the proof. ☐

Moreover, to obtain the discrete energy estimates, we state in the following lemma some key identities. The proof can be found in [38].

Lemma 5. *[38] For all $K,\ L \in \mathcal{T}$. We have the following inequality:*

$$M_{\mathrm{w},KL} + M_{\mathrm{nw},KL} \ge M_0,$$

and

$$M_0 (p_K^n - p_L^n)^2 + (\mathcal{B}(s_{\mathrm{w},K}^n) - \mathcal{B}(s_{\mathrm{w},L}^n))^2 \le \sum_{\alpha\in\{\mathrm{w},\mathrm{nw}\}} M_{\alpha,KL}(p_{\alpha,K} - p_{\alpha,L})^2.$$

Proposition 4. *Let $\mathscr{D}$ be a space time discretization as defined in Section 5.1, $(\boldsymbol{u}_{\mathscr{D}}, p_{\mathrm{w},\mathscr{D}}, p_{\mathrm{nw},\mathscr{D}}, T_{\mathscr{D}})$ be the solution of* (5.20)*–*(5.23) *and $\alpha \in \{\mathrm{w},\mathrm{nw}\}$. Assume that the assumptions stated in Section 2.2 are satisfied. We have the following estimates:*

$$
\begin{aligned}
&\|\partial_t^n \boldsymbol{u}_{\mathscr{D}}\|_{L^\infty(0,t_F;(L^2(\Omega))^d)} + \|p_{\alpha,\mathscr{D}}\|_{L^\infty(0,t_F;L^2(\Omega))} + \|T_{\mathscr{D}}\|_{L^\infty(0,t_F;L^2(\Omega))} \\
&+ \|e_{\alpha,\mathscr{D}}\|_{L^\infty(0,t_F;L^1(\Omega))} + \|\boldsymbol{u}_{\mathscr{D}}\|_{L^\infty(0,t_F;(H^1(\Omega))^d)} + \|\boldsymbol{\nabla}_h p_{\mathscr{D}}\|_{L^2(0,t_F;(L^2(\Omega))^d)} \\
&+ \|\boldsymbol{\nabla}_h T_{\mathscr{D}}\|_{L^2(0,t_F;(L^2(\Omega))^d)} + \|U_{\mathscr{D}}\|_{L^\infty(0,t_F;L^1(\Omega))} + \|\pi_{\mathscr{D}}\|_{L^\infty(0,t_F;L^2(\Omega))} \le C.
\end{aligned}
$$

Proof of Proposition 4. First, we sum inequality (6.3) over $n = 1,\dots,N_T$ to obtain

$$
\begin{aligned}
&\int_\Omega \left(E_h^{N_T} - E_h^0 \right) \mathrm{d}\boldsymbol{x} + \int_\Omega \frac{m_0}{2} \left(\left|\partial_t^{N_T}\boldsymbol{u}_h\right|^2 - \left|\partial_t^0 \boldsymbol{u}_h\right|^2 \right) \mathrm{d}\boldsymbol{x} \\
&+ \frac{\delta t}{d} \sum_{n=1}^{N_T} \int_\Omega k \left(M_{\mathrm{w},h}^n |\boldsymbol{\nabla}_h p_{\mathrm{w},h}^n|^2 + M_{\mathrm{nw},h}^n |\boldsymbol{\nabla}_h p_{\mathrm{nw},h}^n|^2 \right) \mathrm{d}\boldsymbol{x} + \frac{\delta t}{d} \sum_{n=1}^{N_T} \int_\Omega \frac{\lambda}{T_{\mathrm{ref}}} |\boldsymbol{\nabla}_h T_h^n|^2 \,\mathrm{d}\boldsymbol{x} \\
&\le \delta t \sum_{n=1}^{N_T} \int_\Omega \left(\sum_{\alpha\in\{\mathrm{w},\mathrm{nw}\}} r_{\alpha,h}^n \left(\frac{p_{\alpha,h}^n}{\rho_\alpha} + e_{\alpha,h}^n - c_{\alpha,h}^n T_h^n \right) + r_{e,h}^n + \boldsymbol{f}_{u,h}^n \cdot \partial_t^n \boldsymbol{u}_h \right) \mathrm{d}\boldsymbol{x}.
\end{aligned}
\tag{6.19}
$$

Let $\{y^n\}_{i=1}^\infty$ be the sequence defined by

$$y^n := \int_\Omega \left(E_h^n + \frac{m_0}{2} |\partial_t^n \boldsymbol{u}_h|^2 \right) \mathrm{d}\boldsymbol{x}.$$

Our goal is to apply the discrete Gronwall Lemma. To that end, we aim to show that there exist two sequences $\{a^n\}_{n=1}^\infty$ and $\{b^n\}_{n=1}^\infty$ such that $y^n \le a^n + \sum_{i=1}^{n-1} b^i y^i$, which is the required form to apply

Gronwall's inequality and deduce the desired bounds. Now, using the definition of y^n and employing inequality (6.19), we obtain

$$\begin{aligned} & y^n - y^0 \\ & \leq \int_\Omega \left(E_h^n - E_h^0\right) \mathrm{d}\boldsymbol{x} + \int_\Omega \frac{m_0}{2} \left(\left|\partial_t^n \boldsymbol{u}_h\right|^2 - \left|\partial_t^0 \boldsymbol{u}_h\right|^2\right) \mathrm{d}\boldsymbol{x} \\ & + \frac{\delta t}{d} \sum_{i=1}^n \int_\Omega k \left(M_{\mathrm{w},h}^i |\boldsymbol{\nabla}_h p_{\mathrm{w},h}^i|^2 + M_{\mathrm{nw},h}^i |\boldsymbol{\nabla}_h p_{\mathrm{nw},h}^i|^2\right) \mathrm{d}\boldsymbol{x} + \frac{\delta t}{d} \sum_{i=1}^n \int_\Omega \frac{\lambda}{T_{\mathrm{ref}}} |\boldsymbol{\nabla}_h T_h^i|^2 \,\mathrm{d}\boldsymbol{x} \\ & \leq \delta t \sum_{i=1}^n \int_\Omega \left(\sum_{\alpha \in \{\mathrm{w},\mathrm{nw}\}} r_{\alpha,h}^i \left(\frac{p_{\alpha,h}^i}{\rho_\alpha} + e_{\alpha,h}^i - c_{\alpha,h}^i T_h^i \right) + r_{e,h}^i + \boldsymbol{f}_{u,h}^i \cdot \partial_t^i \boldsymbol{u}_h \right) \mathrm{d}\boldsymbol{x}. \end{aligned} \tag{6.20}$$

From Assumptions **(A2)** and **(A3)**, there exist positive constants $(C_{1,\alpha}, C_2, C_3, C_{4,\alpha}) \in \mathbb{R}_+^4$, $\alpha \in \{\mathrm{w},\mathrm{nw}\}$ such that

$$|r_{\alpha,h}^n| \leq C_{1,\alpha},\ |r_{e,h}^n| \leq C_2,\ |\boldsymbol{f}_{u,h}^n| \leq C_3,\ e_{\alpha,h}^n - T_h^n c_{\alpha,h}^n \leq C_{4,\alpha} |T_h^n|^2. \tag{6.21}$$

We proceed analogously to the proof of Proposition 1. Employing inequality (6.21) together with Young's inequality, we obtain the following estimate on the right-hand side of (6.20):

$$\begin{aligned} & \int_\Omega \left(\sum_{\alpha \in \{\mathrm{w},\mathrm{nw}\}} r_{\alpha,h}^n \left(\frac{p_{\alpha,h}^n}{\rho_\alpha} + e_{\alpha,h}^n - c_{\alpha,h}^n T_h^n \right) + r_{e,h}^n + \boldsymbol{f}_{u,h}^n \cdot \partial_t^n \boldsymbol{u}_h \right) \mathrm{d}\boldsymbol{x} \\ & \leq \int_\Omega \left(\left(\frac{C_{1,\mathrm{w}}}{\rho_\mathrm{w}} + \frac{C_{1,\mathrm{nw}}}{\rho_\mathrm{nw}} \right) p_{\mathrm{w},h}^n + (C_{1,\mathrm{w}} C_{4,\mathrm{w}} + C_{1,\mathrm{nw}} C_{4,\mathrm{nw}}) |T_h^n|^2 \right) \mathrm{d}\boldsymbol{x} \\ & + \int_\Omega \left(\frac{C_{1,\mathrm{nw}}}{\rho_\mathrm{nw}} p_\mathrm{c}(s_{\mathrm{w},h}^n) + C_2 + \frac{1}{2} \left(|\boldsymbol{f}_{u,h}^n|^2 + |\partial_t^n \boldsymbol{u}_h|^2 \right) \right) \mathrm{d}\boldsymbol{x} \end{aligned} \tag{6.22}$$

Furthermore, from the discrete equivalent pore pressure π_h^n definition given in (5.13), the fact that $U_h^n \geq 0$, $s_{\alpha,h}^n \in [0,1]$ and Assumption (**A7**), we obtain that $p_{\mathrm{w},h}^n \leq \pi_h^n \leq p_{\mathrm{nw},h}^n$. Consequently, equation (6.22) yields

$$\begin{aligned} & \int_\Omega \left(\sum_{\alpha \in \{\mathrm{w},\mathrm{nw}\}} r_{\alpha,h}^n \left(\frac{p_{\alpha,h}^n}{\rho_\alpha} + e_{\alpha,h}^n - c_{\alpha,h}^n T_h^n \right) + r_{e,h}^n + \boldsymbol{f}_{u,h}^n \cdot \partial_t^n \boldsymbol{u}_h \right) \mathrm{d}\boldsymbol{x} \\ & \leq \int_\Omega \left(C_7 \left(|\pi_h^n|^2 + |T_h^n|^2 \right) + C_8 + \frac{1}{2} |\partial_t^n \boldsymbol{u}_h|^2 \right) \mathrm{d}\boldsymbol{x}, \end{aligned} \tag{6.23}$$

where $C_7 := \max\left(\frac{C_{1,\mathrm{w}}}{2\rho_\mathrm{w}} + \frac{C_{1,\mathrm{nw}}}{2\rho_\mathrm{nw}} + C_{1,\mathrm{w}} C_{4,\mathrm{w}} + C_{1,\mathrm{nw}} C_{4,\mathrm{nw}}, \right)$ and $C_8 := C_2 + \frac{C_{1,\mathrm{w}}}{2\rho_\mathrm{w}} + \frac{C_{1,\mathrm{nw}}}{2\rho_\mathrm{nw}} + \frac{C_{1,\mathrm{nw}}}{\rho_\mathrm{nw}} p_\mathrm{c}(0) + \frac{1}{2} C_3^2$ are positive constants.

Now, using the fact that $\mathbb{M}$ is a real symmetric matrix, we have

$$\lambda_{\min} |(\pi_h^n, T_h^n)|^2 \leq \begin{bmatrix} \pi_h^n & T_h^n \end{bmatrix} \mathbb{M} \begin{bmatrix} \pi_h^n \\ T_h^n \end{bmatrix} \leq \lambda_{\max} |(\pi_h^n, T_h^n)|^2, \tag{6.24}$$

where $\lambda_{\min}$ and $\lambda_{\max}$ denote the minimum and maximum eigenvalues of $\mathbb{M}$, respectively. Moreover, employing Assumptions (**A1**) and (**A4**) together with inequality (6.24) and the fact that $U_h^n \geq 0$, we obtain from the definition of E_h^n

$$\begin{aligned} E_h^n & \geq \frac{1}{2} \lambda_{\min} |(\pi_h^n, T_h^n)|^2 + \rho_\mathrm{w} \phi_* e_{\mathrm{w},h}^n + \rho_\mathrm{nw} \phi_* e_{\mathrm{nw},h}^n \\ & \geq C_0 \left(|\pi_h^n|^2 + |T_h^n|^2 + e_{\mathrm{w},h}^n + e_{\mathrm{nw},h}^n \right), \end{aligned} \tag{6.25}$$

where $C_0 := \min\left(\frac{1}{2} \lambda_{\min}, \rho_\mathrm{w} \phi_*, \rho_\mathrm{nw} \phi_* \right)$ is a real positive constant.

Employing inequalities (6.23) and (6.25) in equation (6.20), we obtain

$$\begin{aligned} y^n &\leq y^0 + \delta t \sum_{i=1}^{n} \int_{\Omega} \left(C_7 \left(|\pi_h^i|^2 + |T_h^i|^2\right) + C_8 + \frac{1}{2}|\partial_t^i \boldsymbol{u}_h|^2 \right) \mathrm{d}\boldsymbol{x} \\ &\leq a + b\delta t \sum_{i=1}^{n} y^i, \end{aligned} \tag{6.26}$$

where $b := \max\left(\frac{C_7}{C_0}, \frac{1}{m_0}\right)$ and $a := y^0 + C_8 t_F |\Omega|$. Now, we apply the discrete Gronwall Lemma to inequality (6.26). It follows that

$$y^n \leq a e^{bn\delta t} \leq a e^{b t_F},\ n = 1, \ldots, N_T.$$

Hence, there exists a constant $C > 0$ such that

$$\int_{\Omega} \left(E_h^n + \frac{m_0}{2}|\partial_t^n \boldsymbol{u}_h|^2 \right) \mathrm{d}\boldsymbol{x} \leq C, \quad \text{for all } n = 1, \ldots, N_T.$$

Therefrom, $E_{\mathscr{D}} \in L^\infty(0, t_F; L^1(\Omega))$ and $\partial_t^n \boldsymbol{u}_{\mathscr{D}} \in L^\infty(0, t_F; (L^2(\Omega))^d)$. Moreover, using the lower bound (6.25) on E_h^n, we obtain that $\pi_{\mathscr{D}} \in L^\infty(0, t_F; L^2(\Omega))$, $T_{\mathscr{D}} \in L^\infty(0, t_F; L^2(\Omega))$, $e_{\mathrm{w},\mathscr{D}} \in L^\infty(0, t_F; L^1(\Omega))$ and $e_{\mathrm{nw},\mathscr{D}} \in L^\infty(0, t_F; L^2(\Omega))$. Furthermore, since $\pi_{\mathscr{D}} - p_{\mathrm{c},\mathscr{D}} \leq p_{\mathrm{w},\mathscr{D}} \leq \pi_{\mathscr{D}}$, then $p_{\mathrm{w},\mathscr{D}} \in L^\infty(0, t_F; L^2(\Omega))$ and $p_{\mathrm{nw},\mathscr{D}} \in L^\infty(0, t_F; L^2(\Omega))$. Additionally, by the definition of E_h^n, we have that $\epsilon(\boldsymbol{u}_{\mathscr{D}}) \in L^\infty(0, t_F; (L^2(\Omega)^{d\times d}))$; therefore, $\boldsymbol{\nabla}_h \boldsymbol{u}_{\mathscr{D}} \in L^\infty(0, t_F; (L^2(\Omega))^{d\times d})$, and it follows that $\boldsymbol{u}_{\mathscr{D}} \in L^\infty(0, t_F; (H^1(\Omega))^d)$.

Moreover, using equations (6.20) and (6.23), we get

$$\frac{\delta t}{d} \sum_{n=1}^{N_T} \int_{\Omega} k \left(M_{\mathrm{w},h}^n |\boldsymbol{\nabla}_h p_{\mathrm{w},h}^n|^2 + M_{\mathrm{nw},h}^n |\boldsymbol{\nabla}_h p_{\mathrm{nw},h}^n|^2 \right) \mathrm{d}\boldsymbol{x} + \frac{\delta t}{d} \sum_{n=1}^{N_T} \int_{\Omega} \frac{\lambda}{T_{\mathrm{ref}}} |\boldsymbol{\nabla}_h T_h^n|^2 \,\mathrm{d}\boldsymbol{x} \leq C$$

Employing the inequalities in Lemma 5, we obtain

$$\frac{\delta t}{d} \sum_{n=1}^{N_T} \int_{\Omega} k \left(M_0 |\boldsymbol{\nabla}_h p_h^n|^2 + |\boldsymbol{\nabla}_h \mathcal{B}(s_{\mathrm{w},h}^n)|^2 \right) \mathrm{d}\boldsymbol{x} \leq C$$

Then, we get that $\boldsymbol{\nabla}_h p_{\mathscr{D}} \in L^\infty(0, t_F; (L^2(\Omega))^d)$, $\boldsymbol{\nabla}_h \mathcal{B}(s_{\mathrm{w},\mathscr{D}}) \in L^\infty(0, t_F; (L^2(\Omega))^d)$ and $\boldsymbol{\nabla}_h T_{\mathscr{D}} \in L^\infty(0, t_F; (L^2(\Omega))^d)$ which concludes the proof. □

7 Convergence analysis

In this section, we present the following theorem stating the convergence of the sequence of discrete solutions to the continuous weak solution.

Theorem 1. *Let the assumption in Section 2.2 hold, $(\mathscr{D}_m)_{m\in\mathbb{N}}$ be a sequence of space time discretizations defined in Section 5.1 such that $\lim_{m\to\infty} \mathrm{size}(\mathscr{D}_m) = 0$. Assume that, for each $m \in \mathbb{N}$, the finite volume scheme (5.2)–(5.8) has a solution $(p_{\mathrm{w},\mathscr{D}_m}, p_{\mathrm{nw},\mathscr{D}_m}, T_{\mathscr{D}_m}, \boldsymbol{u}_{\mathscr{D}_m})_{m\in N}$. Then, $(p_{\mathrm{w},\mathscr{D}_m}, p_{\mathrm{nw},\mathscr{D}_m}, T_{\mathscr{D}_m}, \boldsymbol{u}_{\mathscr{D}_m})_{m\in N}$ converges, up to a subsequence, to the weak solution $(p_{\mathrm{w}}, p_{\mathrm{nw}}, T, \boldsymbol{u})$ of (2.1) in the sense of Definition 1.*

In order to prove this theorem, we follow the approach presented in [38]. First, we state some compactness properties for $\phi_{\mathscr{D}} s_{\alpha,\mathscr{D}}$ and $S_{s,\mathscr{D}} + \sum_{\alpha\in\{\mathrm{w},\mathrm{nw}\}} \phi_{\mathscr{D}} c_{\alpha,\mathscr{D}} s_{\alpha,\mathscr{D}}$, then we study the limits of the discrete variables.

7.1 Compactness properties

In this section, we derive estimates on the space and time translates of the functions $U_{\alpha,\mathscr{D}} := \phi_{\mathscr{D}} s_{\alpha,\mathscr{D}}$ for $\alpha \in \{\mathrm{w}, \mathrm{nw}\}$ and $V_{\mathscr{D}} := S_{s,\mathscr{D}} + \sum_{\alpha\in\{\mathrm{w},\mathrm{nw}\}} \phi_{\mathscr{D}} c_{\alpha,\mathscr{D}} s_{\alpha,\mathscr{D}}$, which implies that $U_{\alpha,\mathscr{D}}$ and $V_{\mathscr{D}}$ are relatively compact in $L^1(Q_{t_F})$ as a result of the Kolmogorov-Riesz-Fréchet theorem [13, Theorem 4.26]. To this end, we replace the study of the discrete functions $\phi_{\mathscr{D}} s_{\alpha,\mathscr{D}}$ and $S_{s,\mathscr{D}} + \sum_{\alpha\in\{\mathrm{w},\mathrm{nw}\}} \phi_{\mathscr{D}} c_{\alpha,\mathscr{D}} s_{\alpha,\mathscr{D}}$ (constant per cylinder $Q_K^n := (t^{n-1}, t^n) \times K$) by the study of the functions $\overline{U}_{\alpha,\mathscr{D}} := \overline{\phi}_{\mathscr{D}} \overline{s}_{\alpha,\mathscr{D}}$ and $\overline{V}_{\mathscr{D}} := \overline{S}_{s,\mathscr{D}} + \sum_{\alpha\in\{\mathrm{w},\mathrm{nw}\}} \overline{\phi}_{\mathscr{D}} \overline{c}_{\alpha,\mathscr{D}} \overline{s}_{\alpha,\mathscr{D}}$. The latter being continuous affine in time and constant in space defined by

$$\overline{U}_{\alpha,\mathscr{D}}(t,\boldsymbol{x}) := \sum_{n=1}^{N_T} \sum_{K\in\mathcal{T}} \frac{1}{\delta t} \left((t-(n-1)\delta t) \phi_K^n s_K^n + (n\delta t - t) \phi_K^{n-1} s_K^{n-1} \right) \mathbb{1}_{Q_K^n}(t,\boldsymbol{x}).$$

and

$$\overline{V}_{\mathscr{D}}(t,\boldsymbol{x}) := \sum_{n=1}^{N_T} \sum_{K\in\mathcal{T}} \frac{1}{\delta t} \left((t-(n-1)\delta t) V_K^n + (n\delta t - t) V_K^{n-1} \right) \mathbb{1}_{Q_K^n}(t,\boldsymbol{x}).$$

Lemma 6. *(Estimates on space translate of $\overline{U}_{\alpha,\mathscr{D}}$ and $\overline{V}_{\mathscr{D}}$). Let the assumptions presented in Section 2.2 be valid. Let $\mathscr{D}$ be a finite volume discretization of $\Omega \times (0, t_F)$ and let $(p_{\mathrm{w},\mathscr{D}}, p_{\mathrm{nw},\mathscr{D}}, T_{\mathscr{D}}, \boldsymbol{u}_{\mathscr{D}}))$ be the solution of* (5.20)*–*(5.23)*. Then, we have the following inequalities*

$$\int_{(0,\infty)\times\Omega'} \left| \overline{U}_{\alpha,\mathscr{D}}(t,\boldsymbol{x}+\boldsymbol{y}) - \overline{U}_{\alpha,\mathscr{D}}(t,\boldsymbol{x}) \right| \,\mathrm{d}\boldsymbol{x}\mathrm{d}t \le \omega(|\boldsymbol{y}|), \tag{7.1}$$

and

$$\int_{(0,\infty)\times\Omega'} \left| \overline{V}_{\mathscr{D}}(t,\boldsymbol{x}+\boldsymbol{y}) - \overline{V}_{\mathscr{D}}(t,\boldsymbol{x}) \right| \,\mathrm{d}\boldsymbol{x}\mathrm{d}t \le \eta(|\boldsymbol{y}|), \tag{7.2}$$

for all $\boldsymbol{y} \in \mathbb{R}^d$ with $\Omega' := \{x \in \Omega, [\boldsymbol{x}, \boldsymbol{x}+\boldsymbol{y}] \subset \Omega\}$ and $\lim\limits_{|\boldsymbol{y}|\to 0} (\omega(|\boldsymbol{y}|), \eta(|\boldsymbol{y}|)) = (0,0)$.

Proof of Lemma 6. From the definition of $U_{\alpha,\mathscr{D}}$, Assumption (**A1**) and Lemma 3, we have thanks to the triangle inequality

$$\begin{aligned} &\int_{(0,t_F)\times\Omega'} |U_{\alpha,\mathscr{D}}(t,\boldsymbol{x}+\boldsymbol{y}) - U_{\alpha,\mathscr{D}}(t,\boldsymbol{x})| \,\mathrm{d}\boldsymbol{x}\mathrm{d}t \\ &\le \int_{(0,t_F)\times\Omega'} |\phi_{\mathscr{D}}(t,\boldsymbol{x}+\boldsymbol{y}) \left(s_{\alpha,\mathscr{D}}(t,\boldsymbol{x}+\boldsymbol{y}) - s_{\alpha,\mathscr{D}}(t,\boldsymbol{x}) \right)| \,\mathrm{d}\boldsymbol{x}\mathrm{d}t \\ &+ \int_{(0,t_F)\times\Omega'} |s_{\alpha,\mathscr{D}}(t,\boldsymbol{x}) \left(\phi_{\mathscr{D}}(t,\boldsymbol{x}+\boldsymbol{y}) - \phi_{\mathscr{D}}(t,\boldsymbol{x}) \right)| \,\mathrm{d}\boldsymbol{x}\mathrm{d}t \le U_1 + U_2, \end{aligned}$$

where U_1 and U_2 are defined as follows

$$U_1 := \int_{(0,t_F)\times\Omega'} |s_{\alpha,\mathscr{D}}(t,\boldsymbol{x}+\boldsymbol{y}) - s_{\alpha,\mathscr{D}}(t,\boldsymbol{x})| \,\mathrm{d}\boldsymbol{x}\mathrm{d}t,$$

$$U_2 := \int_{(0,t_F)\times\Omega'} |\phi_{\mathscr{D}}(t,\boldsymbol{x}+\boldsymbol{y}) - \phi_{\mathscr{D}}(t,\boldsymbol{x})| \,\mathrm{d}\boldsymbol{x}\mathrm{d}t.$$

Moreover, we assume that $\mathcal{B}^{-1}$, the inverse of the Kirchhoff transform $\mathcal{B}$ defined by (2.12), is a Hölder function of order $\theta \in]0,1]$ on $[0, \mathcal{B}(1)]$, i.e, there exists a constant C such that for all $a, b \in [0, \mathcal{B}(1)]$, we have $|\mathcal{B}^{-1}(a) - \mathcal{B}^{-1}(b)| \le C|a-b|^\theta$. Following the same arguments presented in [38, Lemma 7.1]. We have

$$U_1 \le C|\boldsymbol{y}|^\theta, \tag{7.3}$$

where C denotes a generic constant here and throughout the proof.

Now, we aim to obtain an estimate on the spacial translate U_2. From the definition of the discrete porosity given in equation (5.9), we obtain

$$\begin{aligned} &\phi_{\mathscr{D}}(t,\boldsymbol{x}) - \phi_h^0(\boldsymbol{x}) = \\ &b(\mathrm{div}_h\, \boldsymbol{u}_{\mathscr{D}}(t,\boldsymbol{x}) - \mathrm{div}_h\, \boldsymbol{u}_h^0(\boldsymbol{x})) - 3\alpha_\phi (T_{\mathscr{D}}(t,\boldsymbol{x}) - T_h^0) + \frac{1}{N}(\pi_{\mathscr{D}}(t,\boldsymbol{x}) - \pi_h^0(\boldsymbol{x})). \end{aligned} \tag{7.4}$$

Substituting equation (7.4) in U_2, we obtain

$$\begin{aligned}
U_2 &= \int_{(0,t_F)\times\Omega'} |\phi_{\mathscr{D}}(t,\boldsymbol{x}+\boldsymbol{y}) - \phi_{\mathscr{D}}(t,\boldsymbol{x})| \,\mathrm{d}\boldsymbol{x}\mathrm{d}t \\
&\leq \int_{(0,t_F)\times\Omega'} \left|\phi^0_{\mathscr{D}}(\boldsymbol{x}+\boldsymbol{y}) - \phi^0_h(\boldsymbol{x})\right| \,\mathrm{d}\boldsymbol{x}\mathrm{d}t \\
&+ \int_{(0,t_F)\times\Omega'} b\left|\operatorname{div}_h \boldsymbol{u}_{\mathscr{D}}(t,\boldsymbol{x}+\boldsymbol{y}) - \operatorname{div}_h \boldsymbol{u}_{\mathscr{D}}(t,\boldsymbol{x})\right| \,\mathrm{d}\boldsymbol{x}\mathrm{d}t \\
&+ \int_{(0,t_F)\times\Omega'} b\left|\operatorname{div}_h \boldsymbol{u}^0_h(\boldsymbol{x}+\boldsymbol{y}) - \operatorname{div}_h \boldsymbol{u}^0_h(\boldsymbol{x})\right| \,\mathrm{d}\boldsymbol{x}\mathrm{d}t \\
&+ \int_{(0,t_F)\times\Omega'} 3\alpha_\phi\left|T_{\mathscr{D}}(t,\boldsymbol{x}+\boldsymbol{y}) - T_{\mathscr{D}}(t,\boldsymbol{x})\right| \,\mathrm{d}\boldsymbol{x}\mathrm{d}t \\
&+ \int_{(0,t_F)\times\Omega'} 3\alpha_\phi\left|T^0_h(\boldsymbol{x}+\boldsymbol{y}) - T^0_h(\boldsymbol{x})\right| \,\mathrm{d}\boldsymbol{x}\mathrm{d}t \\
&+ \int_{(0,t_F)\times\Omega'} \frac{1}{N}\left|\pi_{\mathscr{D}}(t,\boldsymbol{x}+\boldsymbol{y}) - \pi_{\mathscr{D}}(t,\boldsymbol{x})\right| \,\mathrm{d}\boldsymbol{x}\mathrm{d}t \\
&+ \int_{(0,t_F)\times\Omega'} \frac{1}{N}\left|\pi^0_h(\boldsymbol{x}+\boldsymbol{y}) - \pi^0_h(\boldsymbol{x})\right| \,\mathrm{d}\boldsymbol{x}\mathrm{d}t.
\end{aligned} \tag{7.5}$$

In addition, each term of (7.5) can be written as follows, for $\xi_{\mathscr{D}} \in \{\operatorname{div}_h \boldsymbol{u}_{\mathscr{D}}, T_{\mathscr{D}}, \pi_{\mathscr{D}}\}$

$$\int_{(0,t_F)\times\Omega'} |\xi_{\mathscr{D}}(t,\boldsymbol{x}+\boldsymbol{y}) - \xi_{\mathscr{D}}(t,\boldsymbol{x})| \,\mathrm{d}\boldsymbol{x}\mathrm{d}t \leq \sum_{n=1}^{N_T} \delta t \sum_{\sigma_{KL}} |\xi^n_L - \xi^n_K| \int_{\Omega'} \chi_{\sigma_{KL}}(\boldsymbol{x},\boldsymbol{x}+\boldsymbol{y})\,\mathrm{d}\boldsymbol{x},$$

where $\chi_{\sigma_{KL}}$ is defined by

$$\chi_{\sigma_{KL}} := \begin{cases} 1 & \text{if the line } [\boldsymbol{x},\boldsymbol{x}+\boldsymbol{y}] \text{ intersects } \sigma_{KL},\ K, \text{ and } L, \\ 0 & \text{otherwise.} \end{cases}$$

And, we have that (see [28, Lemma 9.3])

$$\int_{\Omega'} \chi_{\sigma_{KL}}(\boldsymbol{x},\boldsymbol{x}+\boldsymbol{y})\,\mathrm{d}\boldsymbol{x} \leq |\sigma_{KL}||\boldsymbol{y}|.$$

Applying the discrete energy estimates obtained in Proposition 4 together with Assumption (**A9**), we obtain

$$U_2 \leq C|\boldsymbol{y}|. \tag{7.6}$$

Furthermore, combining the estimates on U_1 and U_2 given in (7.3) and (7.6), we obtain

$$\int_{(0,t_F)\times\Omega'} |U_{\alpha,\mathscr{D}}(t,\boldsymbol{x}+\boldsymbol{y}) - U_{\alpha,\mathscr{D}}(t,\boldsymbol{x})| \,\mathrm{d}\boldsymbol{x}\mathrm{d}t \leq C\left(|\boldsymbol{y}|^\theta + |\boldsymbol{y}|\right). \tag{7.7}$$

In addition, we have

$$\begin{aligned}
&\int_0^\infty \int_{\Omega'} \left|\overline{U}_{\alpha,\mathscr{D}}(t,\boldsymbol{x}+\boldsymbol{y}) - \overline{U}_{\alpha,\mathscr{D}}(t,\boldsymbol{x})\right| \,\mathrm{d}\boldsymbol{x}\mathrm{d}t \\
&\leq 2\int_0^{t_F} \int_{\Omega'} |U_{\alpha,\mathscr{D}}(t,\boldsymbol{x}+\boldsymbol{y}) - U_{\alpha,\mathscr{D}}(t,\boldsymbol{x})| \,\mathrm{d}\boldsymbol{x}\mathrm{d}t + 2\delta t \int_{\Omega'_\delta} \left|U^0_{\alpha,h}(\boldsymbol{x})\right| \,\mathrm{d}\boldsymbol{x},
\end{aligned}$$

where $\Omega'_\delta = \{\boldsymbol{x} \in \Omega,\ \operatorname{dist}(\boldsymbol{x},\Omega') < |\delta|\}$. From inequality (7.7), the boundedness of $U^0_{\alpha,h}$ in $L^1(\Omega'_\delta)$ and the assumption that δt goes to zero when $\operatorname{size}(\mathscr{D}) \to 0$, we obtain the uniform estimation on the space translate of $\overline{U}_{\alpha,\mathscr{D}}$ stated in (7.1).

Similarly, from the definition of $V_{\mathscr{D}}$, Assumption (**A1**) and Lemma 3, we have

$$\begin{aligned}
&\int_{(0,t_F)\times\Omega'} |V_{\mathscr{D}}(t,\boldsymbol{x}+\boldsymbol{y}) - V_{\mathscr{D}}(t,\boldsymbol{x})| \,\mathrm{d}\boldsymbol{x}\mathrm{d}t \\
&\leq \int_{(0,t_F)\times\Omega'} |S_{s,\mathscr{D}}(t,\boldsymbol{x}+\boldsymbol{y}) - S_{s,\mathscr{D}}(t,\boldsymbol{x})| \,\mathrm{d}\boldsymbol{x}\mathrm{d}t \\
&+ \sum_{\alpha\in\{\mathrm{w},\mathrm{nw}\}} \rho_\alpha \int_{(0,t_F)\times\Omega'} |c_{\alpha,\mathscr{D}}(t,\boldsymbol{x}+\boldsymbol{y})\left((\phi_{\mathscr{D}} s_{\alpha,\mathscr{D}})(t,\boldsymbol{x}+\boldsymbol{y}) - (\phi_{\mathscr{D}} s_{\alpha,\mathscr{D}})(t,\boldsymbol{x})\right)| \,\mathrm{d}\boldsymbol{x}\mathrm{d}t \\
&+ \sum_{\alpha\in\{\mathrm{w},\mathrm{nw}\}} \int_{(0,t_F)\times\Omega'} |(\phi_{\mathscr{D}} s_{\alpha,\mathscr{D}})(t,\boldsymbol{x})\left(c_{\alpha,\mathscr{D}}(t,\boldsymbol{x}+\boldsymbol{y}) - c_{\alpha,\mathscr{D}}(t,\boldsymbol{x})\right)| \,\mathrm{d}\boldsymbol{x}\mathrm{d}t \\
&:= V_1 + \sum_{\alpha\in\{\mathrm{w},\mathrm{nw}\}} V_{2,\alpha} + \sum_{\alpha\in\{\mathrm{w},\mathrm{nw}\}} V_{3,\alpha}.
\end{aligned}$$

The term V_1 is treated in the same way as U_2. From the definition of the discrete skeleton entropy given in (5.9), we have

$$S_{s,\mathscr{D}} - S^0_{s,h} = 3\alpha_s K_s \mathrm{div}_h\left(\boldsymbol{u}_{\mathscr{D}} - \boldsymbol{u}^0_h\right) - 3\alpha_\phi(\pi_{\mathscr{D}} - \pi^0_h) + \frac{C_s}{T_{\mathrm{ref}}}(T_{\mathscr{D}} - T^0_h).$$

Then, applying the discrete energy estimates obtained in Proposition 4 together with Assumption (**A9**), we obtain

$$V_1 \leq C|\boldsymbol{y}|. \tag{7.8}$$

Furthermore, from the assumption of small temperature variations around the reference temperature $T_{\mathrm{ref}} > 0$, we get that the discrete temperature $T_{\mathscr{D}}$ is bounded. Using the thermodynamic relation $de_\alpha = T dc_\alpha$ and the linear dependence of e_α on T, we obtain that

$$\frac{dc_\alpha}{dT} = \frac{1}{T}\frac{de_\alpha}{dT} = \frac{k_\alpha}{T},$$

where k_α is a constant. Hence, $c_{\alpha,\mathscr{D}} = c_\alpha(T_{\mathscr{D}})$ is bounded and Lipschitz continuous. Using this fact together with Lemma 6 and inequality (7.7), we obtain

$$\begin{aligned}
V_{2,\alpha} &= \int_{(0,t_F)\times\Omega'} |c_{\alpha,\mathscr{D}}(t,\boldsymbol{x}+\boldsymbol{y})\left((\phi_{\mathscr{D}} s_{\alpha,\mathscr{D}})(t,\boldsymbol{x}+\boldsymbol{y}) - (\phi_{\mathscr{D}} s_{\alpha,\mathscr{D}})(t,\boldsymbol{x})\right)| \,\mathrm{d}\boldsymbol{x}\mathrm{d}t \\
&\leq C \int_{(0,t_F)\times\Omega'} |(\phi_{\mathscr{D}} s_{\alpha,\mathscr{D}})(t,\boldsymbol{x}+\boldsymbol{y}) - (\phi_{\mathscr{D}} s_{\alpha,\mathscr{D}})(t,\boldsymbol{x})| \,\mathrm{d}\boldsymbol{x}\mathrm{d}t \\
&\leq C\left(|\boldsymbol{y}|^\theta + |\boldsymbol{y}|\right).
\end{aligned} \tag{7.9}$$

Moreover, using that $c_{\alpha,\mathscr{D}}$ is lipschitz continuous and $T_{\mathscr{D}} \in L^2(0,t_F;H^1(\Omega))$, we obtain

$$\begin{aligned}
V_{3,\alpha} &\leq C \int_{(0,t_F)\times\Omega'} |c_{\alpha,\mathscr{D}}(t,\boldsymbol{x}+\boldsymbol{y}) - c_{\alpha,\mathscr{D}}(t,\boldsymbol{x})| \,\mathrm{d}\boldsymbol{x}\mathrm{d}t \\
&\leq C \int_{(0,t_F)\times\Omega'} |T_{\mathscr{D}}(t,\boldsymbol{x}+\boldsymbol{y}) - T_{\mathscr{D}}(t,\boldsymbol{x})| \,\mathrm{d}\boldsymbol{x}\mathrm{d}t \\
&\leq C \sum_{n=1}^{N_T} \delta t \sum_{\sigma_{KL}} |T^n_L - T^n_K| \int_{\Omega'} \chi_{\sigma_{KL}}(\boldsymbol{x},\boldsymbol{x}+\boldsymbol{y}) \,\mathrm{d}\boldsymbol{x} \\
&\leq C|\boldsymbol{y}|.
\end{aligned} \tag{7.10}$$

Furthermore, combining the estimates on V_1, $V_{2,\alpha}$ and $V_{3,\alpha}$ given in (7.8), (7.9) and (7.10), we obtain

$$\int_{(0,t_F)\times\Omega'} |V_{\mathscr{D}}(t,\boldsymbol{x}+\boldsymbol{y}) - V_{\mathscr{D}}(t,\boldsymbol{x})| \,\mathrm{d}\boldsymbol{x}\mathrm{d}t \leq C\left(|\boldsymbol{y}|^\theta + |\boldsymbol{y}|\right). \tag{7.11}$$

In addition, we have

$$\begin{aligned}
&\int_0^\infty \int_{\Omega'} \left|\overline{V}_{\mathscr{D}}(t,\boldsymbol{x}+\boldsymbol{y}) - \overline{V}_{\mathscr{D}}(t,\boldsymbol{x})\right| \mathrm{d}\boldsymbol{x}\mathrm{d}t \\
&\leq 2\int_0^{t_F} \int_{\Omega'} \left|V_{\mathscr{D}}(t,\boldsymbol{x}+\boldsymbol{y}) - V_{\mathscr{D}}(t,\boldsymbol{x})\right| \mathrm{d}\boldsymbol{x}\mathrm{d}t + 2\delta t \int_{\Omega'_\delta} \left|V_h^0(\boldsymbol{x})\right| \mathrm{d}\boldsymbol{x},
\end{aligned}$$

where $\Omega'_\delta = \{\boldsymbol{x} \in \Omega,\ \mathrm{dist}(\boldsymbol{x}, \Omega') < |\delta|\}$. From inequality (7.11), the boundedness of V_h^0 in $L^1(\Omega'_\delta)$ and the assumption that δt goes to zero when $\mathrm{size}(\mathscr{D}) \to 0$, we obtain the uniform estimation on the space translate of $\overline{V}_{\mathscr{D}}$ stated in (7.2). □

Furthermore, we state without proof the following lemma on the time translate of $\overline{U}_{\alpha,\mathscr{D}}$ and $\overline{V}_{\mathscr{D}}$. The proof follows the general idea of [3, Lemma A.1].

Lemma 7. *(Estimates on time translate of $\overline{U}_{\alpha,\mathscr{D}}$ and $\overline{V}_{\mathscr{D}}$). Let the assumptions of Section 2.2 hold true. Let $\mathscr{D}$ be the discretization of $\Omega \times (0, t_F)$ as defined in Section 5.1 and let $(p_{\mathrm{w},\mathscr{D}}, p_{\mathrm{nw},\mathscr{D}}, T_{\mathscr{D}}, \boldsymbol{u}_{\mathscr{D}})$ be the solution of* (5.20)–(5.23)*. Then, we have the following inequalities*

$$\int_{(0,t_F-\tau)\times\Omega} \left|\overline{U}_{\alpha,\mathscr{D}}(t+\tau,\boldsymbol{x}) - \overline{U}_{\alpha,\mathscr{D}}(t,\boldsymbol{x})\right| \mathrm{d}\boldsymbol{x}\mathrm{d}t \leq \tilde{\omega}(\tau),$$

and

$$\int_{(0,t_F-\tau)\times\Omega} \left|\overline{V}_{\mathscr{D}}(t+\tau,\boldsymbol{x}) - \overline{V}_{\mathscr{D}}(t,\boldsymbol{x})\right| \mathrm{d}\boldsymbol{x}\mathrm{d}t \leq \tilde{\eta}(\tau),$$

for all $\tau \in (0, t_F)$ and $(\tilde{\omega}(\tau), \tilde{\eta}(\tau)) \to (0,0)$ when $\tau \to 0$.

7.2 Study of the limit

Proposition 5. *Let $(\mathscr{D}_m)_m$ be a sequence of finite volume discretizations of $\Omega \times (0, t_F)$ such that $\lim_{m\to 0} \mathrm{size}(\mathscr{D}_m) = 0$ and assume that the assumptions of Section 2.2 are fulfilled. Then, there exist subsequences, still denoted $(s_{\alpha,\mathscr{D}_m})_{m\in\mathbb{N}}$, $(p_{\alpha,\mathscr{D}_m})_{m\in\mathbb{N}}$, $(T_{\mathscr{D}_m})_{m\in\mathbb{N}}$ and $(\boldsymbol{u}_{\mathscr{D}_m})_{m\in\mathbb{N}}$ verifying the following convergence*

$$\begin{array}{ll}
\left\|U_{\alpha,\mathscr{D}_m} - \overline{U}_{\alpha,\mathscr{D}_m}\right\|_{L^1(\Omega')} \to 0,\ \left\|V_{\mathscr{D}_m} - \overline{V}_{\mathscr{D}_m}\right\|_{L^1(\Omega')} \to 0 & \\
U_{\alpha,\mathscr{D}_m} \to U_\alpha,\ V_{\mathscr{D}_m} \to V & \textit{strongly in } L^p(Q_{t_F}) \textit{ and a.e. in } Q_{t_F} \textit{ for all } p \geq 1, \\
s_{\alpha,\mathscr{D}_m} \to s_\alpha & \textit{strongly in } L^1(Q_{t_F}) \textit{ and a.e. in } Q_{t_F}, \\
p_{\alpha,\mathscr{D}_m} \rightharpoonup p_\alpha,\ T_{\mathscr{D}_m} \rightharpoonup T & \textit{weakly in } L^2(0,t_F;L^2(\Omega)), \\
\boldsymbol{\nabla}_h p_{\mathscr{D}_m} \rightharpoonup \boldsymbol{\nabla} p,\ \boldsymbol{\nabla}_h \mathcal{B}(s_{\mathrm{w},\mathscr{D}_m}) \rightharpoonup \boldsymbol{\nabla}\mathcal{B}(s_{\mathrm{w}}) & \textit{weakly in } L^2(0,t_F;L^2(\Omega)^d), \\
\boldsymbol{\nabla}_h T_{\mathscr{D}_m} \rightharpoonup \boldsymbol{\nabla} T & \textit{weakly in } L^2(0,t_F;L^2(\Omega)^d), \\
\boldsymbol{u}_{\mathscr{D}_m} \rightharpoonup \boldsymbol{u} & \textit{weakly in } L^2(0,t_F;L^2(\Omega)^d), \\
\epsilon(\boldsymbol{u})_{\mathscr{D}_m} \rightharpoonup \epsilon(\boldsymbol{u}) & \textit{weakly in } L^2(0,t_F;(L^2(\Omega)^{d\times d})), \\
\mathrm{div}_h\,(\boldsymbol{u}_{\mathscr{D}_m}) \rightharpoonup \mathrm{div}\,(\boldsymbol{u}) & \textit{weakly in } L^2(0,t_F;L^2(\Omega)), \\
\partial_t \boldsymbol{u}_{\mathscr{D}_m} \rightharpoonup \partial_t \boldsymbol{u} & \textit{weakly in } L^2(0,t_F;L^2(\Omega)^d), \\
\pi_{\mathscr{D}_m} \rightharpoonup \pi,\ \phi_{\mathscr{D}_m} \rightharpoonup \phi,\ S_{s,\mathscr{D}_m} \rightharpoonup S_s & \textit{weakly in } L^2(0,t_F;L^2(\Omega)), \\
U_\alpha = s_\alpha \phi,\ V = S_s + \displaystyle\sum_{\alpha\in\{\mathrm{w},\mathrm{nw}\}} \rho_\alpha c_\alpha s_\alpha \phi & \textit{a.e. in } Q_{t_F},
\end{array}$$

where

$$\begin{aligned}
\pi &= s_{\mathrm{w}} p_{\mathrm{w}} + p_{\mathrm{nw}} s_{\mathrm{nw}} - \int_{s_{\mathrm{w}}}^1 p_{\mathrm{c}}(z)\,\mathrm{d}z, \\
\phi &= \phi^0 + b\,\mathrm{div}\,(\boldsymbol{u} - \boldsymbol{u}^0) - 3\,\alpha_\phi(T - T^0) + \frac{1}{N}(\pi - \pi^0), \\
S_s &= S_s^0 + 3\,\alpha_s K_s \mathrm{div}\,(\boldsymbol{u} - \boldsymbol{u}^0) - 3\,\alpha_\phi(\pi - \pi^0) + \frac{C_s}{T_{\mathrm{ref}}}(T - T^0).
\end{aligned}$$

Proof of Proposition 5. First, we show that $\left\|U_{\alpha,\mathscr{D}_m} - \overline{U}_{\alpha,\mathscr{D}_m}\right\|_{L^1(\Omega')} \to 0$ when $\text{size}(\mathscr{D}_m) \to 0$. To this end, we employ the following convex inequality, for all a, $b \in \mathbb{R}$, $\theta \in [0,1]$,

$$\int_0^1 |\theta a + (1-\theta) b| \, d\theta \geq \frac{1}{2}\left(|a| + |b|\right).$$

Applying this inequality to $a = U^n_{\alpha,\mathscr{D}_m} - U^{n-1}_{\alpha,\mathscr{D}_m}$, $b = U^{n-1}_{\alpha,\mathscr{D}_m} - U^{n-2}_{\alpha,\mathscr{D}_m}$, from the definition of $\overline{U}_{\alpha,\mathscr{D}}$ we obtain

$$\begin{aligned}&\int_0^{t_F}\int_{\Omega'} \left|U_{\alpha,\mathscr{D}_m}(t,\boldsymbol{x}) - \overline{U}_{\alpha,\mathscr{D}_m}(t,\boldsymbol{x})\right| \mathrm{d}\boldsymbol{x}\mathrm{d}t \\ &\leq 2\int_0^{t_F+\delta t}\int_{\Omega'} \left|\overline{U}_{\alpha,\mathscr{D}_m}(t+\delta t,\boldsymbol{x}) - \overline{U}_{\alpha,\mathscr{D}_m}(t,\boldsymbol{x})\right| \mathrm{d}\boldsymbol{x}\mathrm{d}t,\end{aligned}$$

which tends to zero since $\delta t \to 0$ when $\text{size}(\mathscr{D}_m) \to 0$.

Applying the Riesz-Frechet-Kolmogorov compactness theorem to Lemmas 6 and 7, we obtain that the sequence $(\overline{U}_{\alpha,\mathscr{D}_m})_{m\in\mathbb{N}}$ is relatively compact in $L^1(Q_{t_F})$. Then, there exists a function $U_\alpha \in L^1(Q_{t_F})$ such that $(\overline{U}_{\alpha,\mathscr{D}_m})_{m\in\mathbb{N}}$ converges strongly, up to a subsequence, to U_α in $L^1(Q_{t_F})$. Since $\left\|U_{\alpha,\mathscr{D}_m} - \overline{U}_{\alpha,\mathscr{D}_m}\right\|_{L^1(\Omega')} \to 0$, then $U_{\alpha,\mathscr{D}_m} \to U_\alpha$ strongly in $L^1(Q_{t_F})$ and a.e. in Q_{t_F}. In addition, due to the fact that $U_{\alpha,\mathscr{D}_m}$ is bounded, we obtain the convergence in $L^p(Q_{t_F})$, for all $p \geq 1$. Similarly, we obtain that there exists a function $V \in L^1(Q_{t_F})$ such that $V_{\mathscr{D}_m} \to V$ strongly in $L^p(Q_{t_F})$ and a.e. in Q_{t_F}.

Moreover, using Assumption (**A1**), we can recover estimates on the space and time translates for $s_{\alpha,\mathscr{D}_m}$, then by applying the Riesz-Frechet-Kolmogorov compactness theorem, we obtain that there exists $s_\alpha \in L^1(Q_{t_F})$ such that $s_{\alpha,\mathscr{D}_m}$ converges to s_α strongly in $L^1(Q_{t_F})$ and a.e. in Q_{t_F}.

Now, from the discrete energy estimates Proposition 4, we have that, for all $\alpha \in \{\mathrm{w}, \mathrm{nw}\}$, $(p_{\alpha,\mathscr{D}_m})_{m\in\mathbb{N}}$ is bounded in $L^2(Q_{t_F})$ then there exists a function $p_\alpha \in L^2(Q_{t_F})$ such that $p_{\alpha,\mathscr{D}_m} \rightharpoonup p_\alpha$ weakly in $L^2(Q_{t_F})$. In addition, it follows from Proposition 4 that the sequence $(\boldsymbol{\nabla}_h p_{\mathscr{D}_m})_{m\in\mathbb{N}}$ is bounded in $L^2(Q_{t_F})^d$ and as a consequence of the discrete Poincaré inequality, the sequence $(p_{\mathscr{D}_m})_{m\in\mathbb{N}}$ is bounded in $L^2(Q_{t_F})$. Then, there exist two functions $p \in L^2(Q_{t_F})$ and $\boldsymbol{\psi} \in L^2(Q_{t_F})^d$ such that

$$p_{\mathscr{D}_m} \rightharpoonup p \,\text{weakly in } L^2(Q_{t_F}) \text{ and } \boldsymbol{\nabla}_h p_{,\mathscr{D}_m} \rightharpoonup \boldsymbol{\psi} \,\text{weakly in } L^2(Q_{t_F})^d.$$

We can show that $\boldsymbol{\psi} = \boldsymbol{\nabla} p$ in the sense of distributions. For the detailed proof, we refer to [15]. Similarly, from the discrete energy estimates Proposition 4, we have that $(T_{\mathscr{D}_m})_{m\in\mathbb{N}}$ is bounded in $L^2(Q_{t_F})$ and $(\boldsymbol{\nabla}_h T_{\mathscr{D}_m})_{m\in\mathbb{N}}$ is bounded in $L^2(Q_{t_F})^d$, then there exist a function $T \in L^2(Q_{t_F})$ such that

$$T_{\mathscr{D}_m} \rightharpoonup T \,\text{weakly in } L^2(Q_{t_F}) \text{ and } \boldsymbol{\nabla}_h T_{\mathscr{D}_m} \rightharpoonup \boldsymbol{\nabla} T \,\text{weakly in } L^2(Q_{t_F})^d.$$

Furthermore, we also have that $(\boldsymbol{u}_{\mathscr{D}_m})_{m\in\mathbb{N}}$ is bounded in $L^2(Q_{t_F})^d$ and $(\epsilon(\boldsymbol{u}_{\mathscr{D}_m}))_{m\in\mathbb{N}}$ is bounded in $L^2(Q_{t_F})^{d\times d}$, then there exists two functions $\boldsymbol{u} \in L^2(Q_{t_F})^d$ and $\boldsymbol{\eta} \in L^2(Q_{t_F})^{d\times d}$ such that $\boldsymbol{u}_{\mathscr{D}_m} \rightharpoonup \boldsymbol{u}$ weakly in $L^2(Q_{t_F})^d$ and $\epsilon(\boldsymbol{u}_{\mathscr{D}_m}) \rightharpoonup \boldsymbol{\eta}$ weakly in $L^2(Q_{t_F})^{d\times d}$. It remains to show that $\boldsymbol{\eta} = \epsilon(\boldsymbol{u})$. To do so, it can be shown, analogously to the arguments presented for the convergence of the global pressure, that for all $\boldsymbol{\sigma} \in (C_c^\infty(Q_{t_F}))^{d\times d}$ we have

$$\int_{Q_{t_F}} \epsilon(\boldsymbol{u}_{\mathscr{D}_m}) : \boldsymbol{\sigma} \,\mathrm{d}\boldsymbol{x}\mathrm{d}t + \int_{Q_{t_F}} \boldsymbol{u}_{\mathscr{D}_m} \cdot \mathrm{div}\,(\boldsymbol{\sigma})\,\mathrm{d}\boldsymbol{x}\mathrm{d}t \to 0 \quad \text{as } m \to 0.$$

Since $\mathrm{div}_h\,(\boldsymbol{u}_{\mathscr{D}_m}) = tr(\epsilon(\boldsymbol{u}_{\mathscr{D}_m}))$, we obtain that $\mathrm{div}\,_h(\boldsymbol{u}_{\mathscr{D}_m}) \rightharpoonup \mathrm{div}\,(\boldsymbol{u})$ weakly in $L^2(Q_{t_F})$. Moreover, from the discrete energy estimates, we get that $\partial_t \boldsymbol{u}_{\mathscr{D}_m} \rightharpoonup \partial_t \boldsymbol{u}$ weakly in $L^2(Q_{t_F})^d$.

Now, from the discrete energy estimates, we have that $\pi_{\mathscr{D}_m}$ is bounded in $L^2(0,T;L^2(\Omega))$, then there exists $\pi \in L^2(0,T;L^2(\Omega))$ such that $\pi_{\mathscr{D}}$ converges weakly, up to a subsequence, to π. In addition, from the definition of the equivalent pore pressure π given in (2.7), we have

$$\pi_{\mathscr{D}_m} = p_{\mathrm{w},\mathscr{D}_m} s_{\mathrm{w},\mathscr{D}_m} + p_{\mathrm{nw},\mathscr{D}_m} s_{\mathrm{nw},\mathscr{D}_m} - U_{\mathscr{D}_m},$$

where $U_{\mathscr{D}_m} = U(s_{\mathrm{w},\mathscr{D}_m}) = \int_{s_{\mathrm{w},\mathscr{D}_m}}^1 p_{\mathrm{c}}(z)\,\mathrm{d}z$. From Assumption (**A7**), we have that $p_{\mathrm{c}} : [0,1] \to \mathbb{R}^+$ is continuous on $[0,1]$. Since $s_{\mathrm{w},\mathscr{D}_m}$ converges to s_{w} a.e. in Q_{t_F}, it results that $p_{\mathrm{c}}(s_{\mathrm{w},\mathscr{D}_m})$ converges a.e. in

Q_{t_F} to $p_{\mathrm{c}}(s_{\mathrm{w}})$. Since p_{c} is continuous on $[0,1]$, we obtain from the definition of U that U is continuous on $[0,1]$ and $U_{\mathscr{D}_m} = U(s_{\mathrm{w},\mathscr{D}_m})$ converges a.e. on Q_{t_F} to $U(s_{\mathrm{w}})$. Furthermore, since $p_{\alpha,\mathscr{D}_m}$ converges weakly in $L^2(Q_{t_F})$ to p_α, for $\alpha \in \{\mathrm{w},\mathrm{nw}\}$, we get that $\pi = s_{\mathrm{w}}p_{\mathrm{w}} + p_{\mathrm{nw}}s_{\mathrm{nw}} - U(s_{\mathrm{w}})$.

Furthermore, $\phi_{\mathscr{D}_m} = \sum_{\alpha\in\{\mathrm{w},\mathrm{nw}\}} \phi_{\mathscr{D}_m} s_{\alpha,\mathscr{D}_m}$, then there exists $\phi \in L^2(Q_{t_F})$ such that $\phi_{\mathscr{D}_m} \rightharpoonup \phi$ weakly in $L^2(Q_{t_F})$. Using the definition of the discrete porosity $\phi_{\mathscr{D}_m}$ given in (5.9), the limits of $\pi_{\mathscr{D}_m}, T_{\mathscr{D}_m}$ and $\operatorname{div}_h(\boldsymbol{u}_{\mathscr{D}_m})$, and Assumption (**A9**), we obtain that $\phi = \phi^0 + b\,\operatorname{div}(\boldsymbol{u}-\boldsymbol{u}^0) - 3\alpha_\phi(T-T^0) + \frac{1}{N}(\pi-\pi^0)$.

Similarly, using the definition of the discrete skeleton entropy $S_{s,\mathscr{D}_m}$ together with the weak convergence of $\pi_{\mathscr{D}_m}, T_{\mathscr{D}_m}$ and $\operatorname{div}_h(\boldsymbol{u}_{\mathscr{D}_m})$ and Assumption (**A9**), we obtain that

$$S_{s,\mathscr{D}_m} \rightharpoonup S_s = S_s^0 + 3\alpha_s K_s \operatorname{div}(\boldsymbol{u}-\boldsymbol{u}^0) - 3\alpha_\phi(\pi-\pi^0) + \frac{C_s}{T_{\mathrm{ref}}}(T-T^0).$$

Moreover, since $\phi_{\mathscr{D}_m} \rightharpoonup \phi$ and $s_{\alpha,\mathscr{D}_m} \to s_\alpha$, then $U_\alpha = \phi s_\alpha$. In addition, $c_{\alpha,\mathscr{D}_m} \rightharpoonup c_\alpha$ in $L^2(Q_{t_F})$ since c_α is continuous and $T_{\mathscr{D}_m} \rightharpoonup T$. Then, since $S_{s,\mathscr{D}_m} \rightharpoonup S_s$, ρ_α is a constant and $\phi_{\mathscr{D}_m} s_{\alpha,\mathscr{D}_m} \to \phi s_\alpha$ a.e. in Q_{t_F}, we get that $V = S_s + \sum_{\alpha\in\{\mathrm{w},\mathrm{nw}\}} \rho_\alpha c_\alpha s_\alpha \phi$. ☐

Now, we show the proof of Theorem 1

Proof of Theorem 1. In this proof, we want to show that our numerical scheme is convergent such that the limit verifies the weak formulation given in Definition 1. Let $\varphi_\alpha, \psi \in D([0,t_F)\times\overline{\Omega})$, for $\alpha \in \{\mathrm{w},\mathrm{nw}\}$ and $\boldsymbol{\omega} \in (D([0,t_F)\times\overline{\Omega}))^d$. First, we multiply the discrete wetting fluid mass conservation equation (5.20) by δt, then we sum over $n = 1,\cdots,N_T$ to obtain

$$\sum_{n=1}^{N_T} \delta t \int_\Omega \rho_{\mathrm{w}} \frac{s_{\mathrm{w},h}^n \phi_h^n - s_{\mathrm{w},h}^{n-1}\phi_h^{n-1}}{\delta t} \varphi_{\mathrm{w},h}\,\mathrm{d}\boldsymbol{x} + \frac{1}{d}\sum_{n=1}^{N_T} \delta t \int_\Omega \rho_{\mathrm{w}} k M_{\mathrm{w},h}^n \boldsymbol{\nabla}_h p_{\mathrm{w},h}^n \cdot \boldsymbol{\nabla}_h \varphi_{\mathrm{w},h}\,\mathrm{d}\boldsymbol{x}$$
$$= \sum_{n=1}^{N_T} \delta t \int_\Omega r_{\mathrm{w},h}^n \varphi_{\mathrm{w},h}\,\mathrm{d}\boldsymbol{x},$$

which is equivalent to the following discrete variational formulation

$$\begin{aligned}\int_{Q_{t_F}} \rho_{\mathrm{w}} \partial_t \overline{U}_{\mathrm{w},\mathscr{D}} \varphi_{\mathrm{w},\mathscr{D}}\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t + \frac{1}{d}\int_{Q_{t_F}} \rho_{\mathrm{w}} k M_{\mathrm{w},\mathscr{D}} \boldsymbol{\nabla}_h p_{\mathrm{w},\mathscr{D}} \cdot \boldsymbol{\nabla}_h \varphi_{\mathrm{w},\mathscr{D}}\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \\ = \int_{Q_{t_F}} r_{\mathrm{w},\mathscr{D}} \varphi_{\mathrm{w},\mathscr{D}}\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t.\end{aligned} \tag{7.12}$$

Next, multiplying the discrete non-wetting fluid mass conservation equation (5.21) by δt and summing over $n = 1,\cdots,N_T$ yields the following discrete variational formulation

$$\begin{aligned}\int_{Q_{t_F}} \rho_{\mathrm{nw}} \partial_t \overline{U}_{\mathrm{nw},\mathscr{D}} \varphi_{\mathrm{nw},\mathscr{D}}\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t + \frac{1}{d}\int_{Q_{t_F}} \rho_{\mathrm{nw}} k M_{\mathrm{nw},\mathscr{D}} \boldsymbol{\nabla}_h p_{\mathrm{nw},\mathscr{D}} \cdot \boldsymbol{\nabla}_h \varphi_{\mathrm{nw},\mathscr{D}}\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \\ = \int_{Q_{t_F}} r_{\mathrm{nw},\mathscr{D}} \varphi_{\mathrm{nw},\mathscr{D}}\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t.\end{aligned} \tag{7.13}$$

Moreover, we multiply the discrete entropy conservation equation (5.6) by ψ_K, then we sum over K to obtain

$$\begin{aligned}&\sum_{K\in\mathcal{T}} |K| \frac{1}{\delta t}\left(S_{s,K}^n - S_{s,K}^{n-1}\right)\psi_K \\ &+ \sum_{K\in\mathcal{T}} |K| \frac{1}{\delta t} \sum_{\alpha\in\{\mathrm{w},\mathrm{nw}\}} \left(\rho_\alpha \phi_K^n s_{\alpha,K}^n c_{\alpha,K}^n - \sum_{\alpha\in\{\mathrm{w},\mathrm{nw}\}} \rho_\alpha \phi_K^{n-1} s_{\alpha,K}^{n-1} c_{\alpha,K}^{n-1}\right)\psi_K \\ &+ \sum_{K\in\mathcal{T}} \sum_{L\in\mathcal{N}(K)} \sum_{\alpha\in\{\mathrm{w},\mathrm{nw}\}} \rho_\alpha c_{\alpha,KL}^n k M_{\alpha,KL}^n \tau_{KL}\left(p_{\alpha,K}^n - p_{\alpha,L}^n\right)\psi_K \\ &+ \sum_{K\in\mathcal{T}} \sum_{L\in\mathcal{N}(K)} \frac{\lambda}{T_{\mathrm{ref}}} \tau_{KL}\left(T_K^n - T_L^n\right)\psi_K = \sum_{K\in\mathcal{T}} |K| \frac{r_{e,K}^n}{T_K^n}\psi_K.\end{aligned}$$

From the definition of the discrete functions, we have

$$
\begin{aligned}
&\sum_{K\in\mathcal{T}} |K| \frac{1}{\delta t}\left(S^n_{s,K} - S^{n-1}_{s,K}\right)\psi_K = \frac{1}{\delta t}\int_\Omega \left(S^n_{s,h} - S^{n-1}_{s,h}\right)\,\mathrm{d}\boldsymbol{x},\\
&\sum_{K\in\mathcal{T}} |K| \frac{1}{\delta t}\left(\rho_\alpha \phi^n_K s^n_{\alpha,K} c^n_{\alpha,K} - \rho_\alpha \phi^{n-1}_K s^{n-1}_{\alpha,K} c^{n-1}_{\alpha,K}\right)\psi_K\\
&= \frac{1}{\delta t}\int_\Omega \left(\rho_\alpha \phi^n_h s^n_{\alpha,h} c^n_{\alpha,h} - \sum_{\alpha\in\{\mathrm{w},\mathrm{nw}\}} \rho_\alpha \phi^{n-1}_h s^{n-1}_{\alpha,h} c^{n-1}_{\alpha,h}\right)\psi_h\,\mathrm{d}\boldsymbol{x},
\end{aligned}
$$

and

$$
\sum_{K\in\mathcal{T}} |K| \frac{r^n_{e,K}}{T^n_K}\psi_K = \int_\Omega \frac{r^n_{e,h}}{T^n_h}\psi_h\,\mathrm{d}\boldsymbol{x}.
$$

Furthermore, integrating by parts and using the definition of the discrete gradient, we get

$$
\begin{aligned}
&\sum_{K\in\mathcal{T}}\sum_{L\in\mathcal{N}(K)} \left(\rho_\alpha c^n_{\alpha,KL} k M^n_{\alpha,KL}\tau_{KL}\left(p^n_{\alpha,K} - p^n_{\alpha,L}\right)\right)\psi_K\\
&= d\sum_{\sigma_{KL}\in\mathcal{E}_h} \left(\rho_\alpha k |\mathcal{D}_{KL}| c^n_{\alpha,KL} M^n_{\alpha,KL}\left(\frac{p^n_{\mathrm{w},L} - p^n_{\mathrm{w},K}}{d_{KL}}\boldsymbol{n}_{KL}\right)\right)\cdot\left(\frac{\psi^n_L - \psi^n_K}{d_{KL}}\right)\boldsymbol{n}_{KL}\\
&= \frac{1}{d}\int_\Omega \left(\rho_\alpha k c^n_{\alpha,h} M^n_{\alpha,h}\boldsymbol{\nabla}_h p^n_{\alpha,h}\right)\cdot\boldsymbol{\nabla}_h\psi_h\,\mathrm{d}\boldsymbol{x},
\end{aligned}
$$

and

$$
\sum_{K\in\mathcal{T}}\sum_{L\in\mathcal{N}(K)} \frac{\lambda}{T_{\mathrm{ref}}}\tau_{KL}\left(T^n_K - T^n_L\right)\psi_K = \frac{1}{d}\int_\Omega \frac{\lambda}{T_{\mathrm{ref}}}\boldsymbol{\nabla}_h T^n_h\cdot\boldsymbol{\nabla}_h\psi_h\,\mathrm{d}\boldsymbol{x}.
$$

Consequently, we obtain

$$
\begin{aligned}
&\frac{1}{\delta t}\int_\Omega \left(S^n_{s,h} + \sum_{\alpha\in\{\mathrm{w},\mathrm{nw}\}} \rho_\alpha \phi^n_h s^n_{\alpha,h} c^n_{\alpha,h} - S^{n-1}_{s,h} - \sum_{\alpha\in\{\mathrm{w},\mathrm{nw}\}} \rho_\alpha \phi^{n-1}_h s^{n-1}_{\alpha,h} c^{n-1}_{\alpha,h}\right)\psi_h\,\mathrm{d}\boldsymbol{x}\\
&+\frac{1}{d}\int_\Omega \left(\sum_{\alpha\in\{\mathrm{w},\mathrm{nw}\}} \rho_\alpha k c^n_{\alpha,h} M^n_{\alpha,h}\boldsymbol{\nabla}_h p^n_{\alpha,h} + \frac{\lambda}{T_{\mathrm{ref}}}\boldsymbol{\nabla}_h T^n_h\right)\cdot\boldsymbol{\nabla}_h\psi_h\,\mathrm{d}\boldsymbol{x} = \int_\Omega \frac{r^n_{e,h}}{T^n_h}\psi_h\,\mathrm{d}\boldsymbol{x}.
\end{aligned}
\tag{7.14}
$$

Moreover, multiplying equation (7.14) by δt and summing over $n = 1,\cdots,N_T$ yields the following discrete variational formulation

$$
\begin{aligned}
&\int_{Q_{t_F}} \partial_t \overline{V}_{\mathscr{D}}\psi_{\mathscr{D}}\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t + \frac{1}{d}\int_{Q_{t_F}} \left(\sum_{\alpha\in\{\mathrm{w},\mathrm{nw}\}} \rho_\alpha k c_{\alpha,\mathscr{D}} M_{\alpha,\mathscr{D}}\boldsymbol{\nabla}_h p_{\alpha,\mathscr{D}}\right)\cdot\boldsymbol{\nabla}_h\psi_{\mathscr{D}}\,\mathrm{d}\boldsymbol{x}\mathrm{d}t\\
&+\frac{1}{d}\int_{Q_{t_F}} \frac{\lambda}{T_{\mathrm{ref}}}\boldsymbol{\nabla}_h T_{\mathscr{D}}\cdot\boldsymbol{\nabla}_h\psi_{\mathscr{D}}\,\mathrm{d}\boldsymbol{x}\mathrm{d}t = \int_{Q_{t_F}} \frac{r_{e,\mathscr{D}}}{T_{\mathscr{D}}}\psi_{\mathscr{D}}\,\mathrm{d}\boldsymbol{x}\mathrm{d}t.
\end{aligned}
\tag{7.15}
$$

Similarly, by multiplying the discrete momentum balance equation (5.23) by δt and summing over $n = 1,\cdots,N_T$, we obtain

$$
\begin{aligned}
&\int_{Q_{t_F}} \partial_{tt}\boldsymbol{u}_{\mathscr{D}}\cdot\boldsymbol{\omega}_{\mathscr{D}}\,\mathrm{d}\boldsymbol{x}\mathrm{d}t + \frac{1}{d}\int_{Q_{t_F}} \frac{\mathcal{E}}{(1+\nu)}\epsilon(\boldsymbol{u}_{\mathscr{D}}) : \epsilon(\boldsymbol{\omega}_h)\,\mathrm{d}\boldsymbol{x}\mathrm{d}t\\
&+\int_{Q_{t_F}} \frac{\mathcal{E}\nu}{(1+\nu)(1-2\nu)}\mathrm{div}_h\,\boldsymbol{u}_{\mathscr{D}}\mathbb{I}_d : \epsilon(\boldsymbol{\omega}_{\mathscr{D}})\,\mathrm{d}\boldsymbol{x}\mathrm{d}t\\
&-\int_{Q_{t_F}} \left(b\pi_{\mathscr{D}} + 3\alpha_s K_s(T_{\mathscr{D}} - T_{\mathrm{ref}})\right)\mathrm{div}_h\left(\boldsymbol{\omega}_{\mathscr{D}}\right)\mathrm{d}\boldsymbol{x}\,\mathrm{d}t = \int_{Q_{t_F}} \boldsymbol{f}_{u,\mathscr{D}}\cdot\boldsymbol{\omega}_{\mathscr{D}}\,\mathrm{d}\boldsymbol{x}\mathrm{d}t.
\end{aligned}
\tag{7.16}
$$

Now, in order to pass to the limit in the first term of (7.12), we integrate by parts to get

$$\int_{Q_{t_F}} \rho_{\mathrm{w}} \partial_t \overline{U}_{\mathrm{w},\mathscr{D}} \varphi_{\mathrm{w},\mathscr{D}} \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t = -\int_{Q_{t_F}} \rho_{\mathrm{w}} \overline{U}_{\mathrm{w},\mathscr{D}} \partial_t \varphi_{\mathrm{w},\mathscr{D}} \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t - \int_{\Omega} \rho_{\mathrm{w}} U^0_{\mathrm{w},h}(\boldsymbol{x}) \varphi^0_{\mathrm{w},h}(\boldsymbol{x}) \,\mathrm{d}\boldsymbol{x}.$$

Due to the strong convergence of $U_{\mathrm{w},\mathscr{D}_m}$ to $U_{\mathrm{w}} = s_{\mathrm{w}}\phi$ and the strong convergence of $\varphi_{\mathrm{w},\mathscr{D}}$ to φ_{w} in $L^2(Q_{t_F})$, we obtain

$$\int_{Q_{t_F}} \rho_{\mathrm{w}} \overline{U}_{\mathrm{w},\mathscr{D}} \partial_t \varphi_{\mathrm{w},\mathscr{D}} \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \to \int_{Q_{t_F}} \rho_{\mathrm{w}}\, \phi(p_{\mathrm{w}}, p_{\mathrm{nw}}, T, \boldsymbol{u})\, s_{\mathrm{w}} \partial_t \varphi_{\mathrm{w}} \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t.$$

Moreover, using that $\phi^0_h s^0_{\mathrm{w},h}$ converges in L^2 to $\phi^0 s^0_{\mathrm{w}}$, we obtain

$$\int_{\Omega} \rho_{\mathrm{w}} U^0_{\mathrm{w},h}(\boldsymbol{x}) \varphi^0_{\mathrm{w},h}(\boldsymbol{x}) \,\mathrm{d}\boldsymbol{x} \to \int_{\Omega} \rho_{\mathrm{w}} \phi^0 s^0_{\mathrm{w}} \varphi^0_{\mathrm{w}}(\boldsymbol{x}) \,\mathrm{d}\boldsymbol{x}.$$

Furthermore, applying Assumption (**A3**) and the strong convergence of $\varphi_{\mathrm{w},\mathscr{D}_m}$, we obtain the following limit on the right-hand side of (7.12)

$$\int_{Q_{t_F}} r_{\mathrm{w},\mathscr{D}} \varphi_{\mathrm{w},\mathscr{D}} \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \to \int_{Q_{t_F}} r_{\mathrm{w}} \varphi_{\mathrm{w}} \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t.$$

For the second term of (7.12), we note that it is defined on the diamonds, thus we will write an equivalent form of this term. First, recall that

$$M_{\mathrm{w},\mathscr{D}} = \sum_{\sigma_{KL} \in \mathcal{D}_{KL}} M^n_{\mathrm{w},KL} \mathbb{1}_{\mathcal{D}_{KL} \times]t^{n-1}, t^n[}.$$

Let $\boldsymbol{x}_{KL} := \theta\, \boldsymbol{x}_K + (1-\theta)\boldsymbol{x}_L$, be some point on the segment $]\boldsymbol{x}_K, \boldsymbol{x}_L[$, for $0 < \theta < 1$, such that

$$\varphi_{\mathrm{w}}(t^n, \boldsymbol{x}_L) - \varphi_{\mathrm{w}}(t^n, \boldsymbol{x}_K) = d_{KL} \boldsymbol{\nabla} \varphi_{\mathrm{w}}(t^n, \boldsymbol{x}_{KL}) \cdot \boldsymbol{n}_{KL}.$$

Denoting by $(\boldsymbol{\nabla}\varphi_{\mathrm{w}})_{\mathscr{D}}$ the function which is constant by diamond and defined as

$$(\boldsymbol{\nabla}\varphi_{\mathrm{w}})_{\mathscr{D}} = \sum_{\sigma_{KL} \in \mathcal{D}_{KL}} \boldsymbol{\nabla}\varphi_{\mathrm{w}}(t^n, \boldsymbol{x}_{KL}) \mathbb{1}_{\mathcal{D}_{KL} \times]t^{n-1}, t^n[},$$

then the second term of (7.12) can be rewritten as follows:

$$\begin{aligned} A_2 &:= \frac{1}{d} \int_{Q_{t_F}} \rho_{\mathrm{w}} k M_{\mathrm{w},\mathscr{D}} \boldsymbol{\nabla}_h p_{\mathrm{w},\mathscr{D}} \cdot \boldsymbol{\nabla}_h \varphi_{\mathrm{w},\mathscr{D}} \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \\ &= \int_{Q_{t_F}} \rho_{\mathrm{w}} k M_{\mathrm{w},\mathscr{D}} \boldsymbol{\nabla}_h p_{\mathrm{w},\mathscr{D}} \cdot (\boldsymbol{\nabla}\varphi_{\mathrm{w}})_{\mathscr{D}} \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t. \end{aligned}$$

Furthermore, we define $\overline{S}_{\mathrm{w},\mathscr{D}_m}$ and $\underline{S}_{\mathrm{w},\mathscr{D}_m}$ by

$$\overline{S}_{\mathrm{w},\mathscr{D}|\mathcal{D}_{KL} \times (t^{n-1}, t^n]} := \max\{s_{\mathrm{w},K}, s_{\mathrm{w},L}\}, \quad \underline{S}_{\mathrm{w},\mathscr{D}|\mathcal{D}_{KL} \times (t^{n-1}, t^n]} := \min\{s_{\mathrm{w},K}, s_{\mathrm{w},L}\}.$$

We also define A_2^* by

$$A_2^* := \int_{Q_{t_F}} \rho_{\mathrm{w}} k M_{\mathrm{w}}(\underline{S}_{\mathrm{w},\mathscr{D}}) \boldsymbol{\nabla}_h p_{\mathrm{w},\mathscr{D}} \cdot (\boldsymbol{\nabla}\varphi_{\mathrm{w}})_{\mathscr{D}} \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t.$$

We will show that A_2^* converges to the desired limit and $A_2^* - A_2$ goes to zero when $h \to 0$. Using that the function B defined by (2.12) is monotone together with the estimate on $\boldsymbol{\nabla}_h \mathcal{B}(s_{\mathrm{w},h})$ and recalling that $|\mathcal{D}_{KL}| = \frac{1}{d} d^2_{KL} \tau_{KL}$, we get

$$\begin{aligned} &\int_{Q_{t_F}} \left| B(\overline{S}_{\mathrm{w},\mathscr{D}}) - B(\underline{S}_{\mathrm{w},\mathscr{D}}) \right|^2 \mathrm{d}\boldsymbol{x}\mathrm{d}t \\ &\leq \sum_{n=0}^{N_T} \delta t \sum_{K \in \mathcal{T}} \sum_{L \in \mathcal{N}(K)} |\mathcal{D}_{KL}| \left(B(s^n_{\mathrm{w},L}) - B(s^n_{\mathrm{w},K}) \right)^2 \\ &\leq C \mathrm{size}(\mathcal{T})^2 \sum_{n=0}^{N_T} \delta t \sum_{K \in \mathcal{T}} \sum_{L \in \mathcal{N}(K)} \frac{|\sigma_{KL}|}{d_{KL}} \left| B(s^n_{\mathrm{w},L}) - B(s^n_{\mathrm{w},K}) \right|^2 \\ &\leq C \mathrm{size}(\mathcal{T})^2. \end{aligned}$$

Since B^{-1} is continuous, we deduce up to a subsequence

$$|\overline{S}_{\mathrm{w},\mathscr{D}} - \underline{S}_{\mathrm{w},\mathscr{D}}| \to 0 \,\text{a.e. on } Q_{t_F}. \tag{7.17}$$

Moreover, we have $\underline{S}_{\mathrm{w},\mathscr{D}_m} \leq s_{\mathrm{w},\mathscr{D}_m} \leq \overline{S}_{\mathrm{w},\mathscr{D}_m}$ and $s_{\mathrm{w},\mathscr{D}_m} \to s_{\mathrm{w}}$ a.e. on Q_{t_F}. Then, due to the continuity of the mobility function M_{w}, we obtain $M_{\mathrm{w}}(\underline{S}_{\mathrm{w},\mathscr{D}_m}) \to M_{\mathrm{w}}(s_{\mathrm{w}})$ a.e. on Q_{t_F} and in $L^p(Q_{t_F})$ for $p < +\infty$. Then, from the strong convergence of $\varphi_{\mathrm{w},\mathscr{D}_m}$ and the weak convergence of $\boldsymbol{\nabla}_h p_{\mathscr{D}_m}$ and $\boldsymbol{\nabla}_h \mathcal{B}(s_{\mathrm{w},\mathscr{D}_m})$, we obtain (we refer to [38, Theorem 3.4] for more details)

$$A_2^{m,*} \longrightarrow \int_{Q_{t_F}} \rho_{\mathrm{w}} k M_{\mathrm{w}}(s_{\mathrm{w}}) \boldsymbol{\nabla} p_{\mathrm{w}} \cdot \boldsymbol{\nabla} \varphi_{\mathrm{w}} \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t.$$

In order to obtain the limit on A_2^m, it remains to show that $\lim\limits_{m\to+\infty} |A_2^m - A_2^{m,*}| = 0$. We remark that

$$\begin{aligned}&|M^n_{\mathrm{w},KL}(p^n_{\mathrm{w},L} - p^n_{\mathrm{w},K}) - M_{\mathrm{w}}(\min(s^n_{\mathrm{w},K}, s^n_{\mathrm{w},L}))(p^n_{\mathrm{w},L} - p^n_{\mathrm{w},K})| \\ &\leq C|s^n_{\mathrm{w},L} - s^n_{\mathrm{w},K}||p^n_{\mathrm{w},L} - p^n_{\mathrm{w},K}|.\end{aligned}$$

Consequently, we get

$$|A_2^m - A_2^{m,*}| \leq C \int_{Q_{t_F}} |s^n_{\mathrm{w},L} - s^n_{\mathrm{w},K}| \boldsymbol{\nabla}_h p_{\mathrm{w},\mathscr{D}_m} \cdot (\boldsymbol{\nabla}\varphi_{\mathrm{w}})_{\mathscr{D}_m} \,\mathrm{d}\boldsymbol{x}\mathrm{d}t.$$

Applying the Cauchy-Schwarz inequality, the uniform bound on $\boldsymbol{\nabla}_h p_{\mathrm{w},\mathscr{D}_m}$ and the convergence (7.17), we get that $|A_2^m - A_2^{m,*}| \to 0$ when $m \to +\infty$.

Using similar arguments, we obtain the following equation by passing to the limit in equation (7.13)

$$\begin{aligned}&-\int_{Q_{t_F}} \rho_{\mathrm{nw}}\,\phi\, s_{\mathrm{nw}} \partial_t \varphi_{\mathrm{nw}} \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t - \int_{\Omega} \rho_{\mathrm{nw}} \phi^0 s^0_{\mathrm{nw}} \varphi^0_{\mathrm{nw}}(\boldsymbol{x}) \,\mathrm{d}\boldsymbol{x} \\ &+ \int_{Q_{t_F}} \rho_{\mathrm{nw}} k M_{\mathrm{nw}}(s_{\mathrm{nw}}) \boldsymbol{\nabla} p_{\mathrm{nw}} \cdot \boldsymbol{\nabla}\varphi_{\mathrm{nw}} \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t = \int_{Q_{t_F}} r_{\mathrm{nw}} \varphi_{\mathrm{nw}} \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t.\end{aligned}$$

In addition, by passing to the limit in equation (7.15), we obtain the weak formulation (4.3). Finally, using the strong convergence of $\boldsymbol{w}_{\mathscr{D}_m}$ together with weak convergences of $\partial_t \boldsymbol{u}_{\mathscr{D}_m}$, $\epsilon(\boldsymbol{u})_{\mathscr{D}_m}$, $\mathrm{div}_h\, \boldsymbol{u}_{\mathscr{D}_m}$, $\pi_{\mathscr{D}_m}$ and $T_{\mathscr{D}_m}$, we obtain the weak formulation (4.4) by passing to the limit in equation (7.16), which concludes the proof. □

Acknowledgments

This work has received funding from the Région Pays de la Loire. The second author gratefully acknowledges the support of the NASINE project (2023-2025), funded by the PULSAR Academy of Young Researchers in Pays de la Loire.

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