Convergence Analysis of a Finite Volume Scheme for Periodic Numerical Solutions of the Monodomain Model in Cardiac Electrophysiology

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October 28, 2025

Abstract

Cardiac electrophysiology is a scientific field that studies the propagation of electrical signals through myocardial tissue. From a mathematical standpoint, this electrical activity can be represented by the monodomain model, a simplification of the well-known bidomain model. It consists of a highly nonlinear parabolic partial differential equation (PDE), coupled with an ordinary differential equations (ODE). Understanding the periodic behavior of the solution is essential for analyzing cardiac pathologies and to better capture the underlying electrophysiological processes. Recent works have provided existence and uniqueness results of time-periodic solutions at the continuous level. In this work, we consider the monodomain model and construct a numerical scheme preserving the periodicity of the solution. We employ the cell-centered finite volume method for the space discretization and the implicit Euler scheme for the time discretization. We prove that our periodic numerical solution converges to the weak periodic solution of the continuous problem. We emphasize here on the fact that the periodicity is ensured by the appropriate choice of an initial condition, solution to some nonlinear problem. This constitutes the main originality of our work. Finally, we provide numerical experiments showing the strength of the proposed approach.

Keywords: Monodomain problem; Finite volumes; Periodic solutions; Convergence; Cardiac electrophysiology.

1 Introduction

Cardiac electrophysiology studies the heart's electrical activity. A good understanding of its function allows the diagnosis and treatment of abnormalities in myocardial cells that cause their dysfunction and lead to arrhythmias [39]. To achieve this, a simple and practical way to study cardiac electrophysiology without dealing with real hearts is by means of mathematical models and numerical simulations. Among the existing models, cardiac electrophysiology can be mathematically represented by the widely known bidomain model. This model has been first introduced in 1969 by Schmidt [37] and mathematically formulated in 1978 by Tung [42]. It consists of nonlinear parabolic partial differential equations (PDEs) coupled with an ordinary differential equation (ODE). The bidomain model views the heart as two internal and external regions separated by the cell membrane equipped with its own potential [39, 26]. Although the bidomain model offers a detailed representation of the electrical activity of the heart, it is challenging in terms of mathematical analysis and computational cost. In this context, researchers have also shown significant interests in a simplified version of the bidomain model, known as the monodomain model [39]. This model is derived by identically considering anisotropy ratios between the intra- and extracellular volumes and is preferred in most investigations due to its lower complexity for mathematical analysis and numerical simulations; thus offering a good trade-off between accuracy and computational efficiency [32, 7, 29, 41].

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Several authors have addressed the well-posedness of cardiac electrophysiology models mentioned above. For instance, Bourgault et al. [8] demonstrated the existence and uniqueness of both local strong solutions (via semigroup theory) and global weak solutions of the bidomain model(via compactness arguments and Galerkin a priori estimates) for two types of reformulations using a nonlinear operator and a parabolic variational operator. The results were validated for three classical ionic models: FitzHugh–Nagumo [22], Aliev–Panfilov [1], and Rogers–McCulloch [36]. In the same spirit, Pargaei et al. [34] proved the existence and uniqueness of the bidomain model coupled with Morris–Lecar ionic dynamics, using a Faedo–Galerkin method [40] and compactness arguments, and for the uniqueness, it relies on Gronwall's inequality. We also mention Veneroni [44] who proposed more general ionic models and a complete analysis of the bidomain model. Concerning specifically the monodomain problem, few contributions addressed its well-posedness. We mention for instance Hernández et al. [27] who established the existence and uniqueness of weak and strong solutions for the monodomain model coupled with the Rogers–McCulloch ionic model via a Faedo–Galerkin approach and compactness arguments.

Understanding the periodic behavior of electrical signals of the heart is essential to deal with cardiac pathologies. With this mindset, several studies have been further developed through additional exploration of the periodic character of these cardiac electrophysiology models. Fraguela et al. [23] has provided existence and uniqueness results for time-periodic solutions of the monodomain problem using the Faedo-Galerkin method combined with the Schauder fixed-point theorem [10] under periodic boundary conditions. He identified a specific relation between the ionic parameters and the diffusion parameters, allowing us to easily construct periodic solutions. In this spirit, Giga et al. [24] demonstrated the existence of strong time-periodic solutions for the bidomain model using the Faedo-Galerkin approximation associated with Brouwer's fixed point theorem and a priori estimates. Also, we mention the contributions [28] and [21]. More precisely, in [21] the proof relies on the Faedo-Galerkin method and compactness arguments under suitable initial conditions while in [28] a maximal L^p-regularity theory is employed. These two references focus on several ionic models mentioned earlier : FitzHugh–Nagumo, Aliev–Panfilov, or Rogers–McCulloch.

Unfortunately, computing analytical solutions for the monodomain problem is often complicated, to not say impossible. An alternative is to construct robust numerical methods approximating the sought solution. Among the wide range of numerical methods for solving PDEs, we mention the classical finite element method [9, 12], the finite volume method [20], the discontinuous Galerkin method [17, 35] and hybrid high-order methods [18, 16].

Concerning mathematical models in cardiac electrophysiology such as the monodomain model, several surveys [11, 30] employed the finite difference method which is a relatively simple approach that works very well on structured meshes. However, it has significant limitations when applied to complex geometries (such as the heart) and to more sophisticated PDEs. The finite element and the finite volume methods are more flexible and suitable to those challenges. First, let us mention the contributions of [34, 38], both of which employ linear finite elements. Further, in [2] is provided a high-order finite element discretization with a posteriori error estimates including an adaptive stopping criterion for nonlinear solvers.

Next, Bendahmane et al. [4, 3] examined finite volume schemes for the monodomain and bidomain models, proving existence and uniqueness of the discrete solution. Convergence results are also discussed in their work. In particular, their numerical experiments showed convergence rates slightly greater than for the first order. Coudière et al. [13] investigated the stability, convergence, and error estimates of the finite volume method for the monodomain model for unstructured 3D meshes. Furthermore, they also introduced a Discrete Duality Finite Volume (DDFV) method (see [14]), to solve heterogeneous and anisotropic elliptic equations on general unstructured meshes, ensuring that the symmetry and well-posedness properties were maintained. Harrild and Henriquez [25] simulated cardiac activity in anatomically realistic geometries with complex fibre orientations using a finite volume scheme that allows the incorporation of arbitrary geometries.

So far, the existence of a discrete periodic solution for the monodomain model has not been established to the best of our knowledge. The main difficulty lies in the choice of the discrete initial condition that guarantees the time periodicity along the simulation. This is the main contribution of the present work.

Therefore, in this work, after giving in Sections 1 and 2 an introduction, a background, and some necessary preliminary definitions, we consider the monodomain model in one dimension and propose a numerical approach to construct a structure preserving time-periodic solution. More precisely, in Section 3, we employ the cell-centered finite volume method for the space discretization and the implicit Euler scheme for the time discretization. The resulting discrete scheme is then written into its discrete variational problem. We established the existence and uniqueness of the approximate solution under an appropriate condition on the time step. Section 4 and Section 5 are devoted to constructing a sequence of numerical periodic solutions and establishing its convergence to a weak periodic solution of the continuous problem by deriving appropriate energy estimates and compactness arguments. We emphasize here the importance of seeking an initial condition, solution to some nonlinear problem, that guarantees the periodic nature of the solution. thanks to Brouwer's fixed-point theorem. This convergence result constitutes the main originality of our work. Section 6 provides a numerical approximation of the initial condition by using Newton's algorithm. Further, we establish in Section 7 numerical experiments for the three classical ionic models: FitzHugh–Nagumo, Aliev–Panfilov, or Rogers–McCulloch, showing the strength of the proposed approach. Finally,

2 Background and definitions

In this section, we introduce the one-dimensional formulation of the monodomain model, along with some definitions and assumptions used throughout the paper.

Let $\mathcal{I}_T := (0,T) \times (0,L)$ be the space-time domain, where T is the time period and L the spatial length . We consider the following monodomain problem in its one-dimensional setting:

$$\begin{aligned}
&C\partial_t u + f(u,w) - \partial_x\left(\sigma(x)\partial_x u\right) = 0, \quad (t,x) \in (0,\infty)\times(0,L),\\
&\partial_t w = g(u,w), \quad (t,x) \in (0,\infty)\times(0,L),\\
&\sigma(0)\ \partial_t u(t,0) = 0,\\
&\sigma(L)\ \partial_t u(t,L) = s(t),
\end{aligned} \tag{2.1}$$

with initial conditions

$$u(0,x) = u_0, \qquad w(0,x) = w_0. \tag{2.2}$$

Let u denote the transmembrane potential and w be the activation variable . The transmembrane ionic current is modeled by a nonlinear function f while channel dynamics is described by a function g. The conductivity is determined by a positive definite symmetric tensor σ , and $C > 0$ denotes membrane capacitance. We impose Neumann boundary conditions, with $s \in L^\infty(0,+\infty)$ representing a T-periodic stimulus. The initial conditions are given by u_0 and w_0.

We consider $L^2(\mathcal{I})$, the Hilbert space of square summable functions on $\mathcal{I}$ endowed with the L^2 scalar product $(\cdot,\cdot)$. Let $V := H^1(\mathcal{I})$ be the space of L^2 functions on the domain $\mathcal{I}$ which admit a weak gradient in $[L^2(\mathcal{I})]^2$. We denote the dual space of V by V' with the duality pairing $\langle\cdot,\cdot\rangle$ defined by $\langle v,h\rangle_{V'\times V}$

In addition, as a result of the Sobolev embedding theorem in [19] we have

$$V \subset L^p(\mathcal{I}) \subset L^2(\mathcal{I}) \subset L^{p'}(\mathcal{I}) \subset V', \quad \text{for all} \ \ 2 \le p \le 6,$$

with dense and continuous embedding (Gelfand triple).

In the sequel, we provide important assumptions for the well-posedness of the monodomain problem. These assumptions are already given in [8, Lemma 25]. We recall them for the sake of clarity.

Assumption A_1**.** *σ is a continuous function defined on $\mathcal{I}$. There exist two positive constants σ_* and σ^* such that*

$$0 < \sigma_* \le \sigma(x) \le \sigma^*, \qquad \textit{for all a.e. } x \in \mathcal{I}.$$

Assumption A_2 (Structural properties of the source terms f and g (see [8]))**.** *The functions f and g are affine with respect to w and verify*

$$f(u,w) = f_1(u) + f_2(u)w, \quad g(u,w) = g_1(u) + g_2 w$$

where $f_1 : \mathbb{R}\to\mathbb{R}$, $f_2 : \mathbb{R}\to\mathbb{R}$, and $g_1 : \mathbb{R}\to\mathbb{R}$ are continuous functions, and $g_2 \in \mathbb{R}$. We also assume that there exist positive constants $c_j \ge 0$, $j = 1\dots6$ such that $\forall u \in \mathbb{R}$, for all $2 \le p \le 6$,

$$\begin{aligned}
|f_1(u)| &\le c_1 + c_2|u|^{p-1},\\
|f_2(u)| &\le c_3 + c_4|u|^{p/2-1},\\
|g_1(u)| &\le c_5 + c_6|u|^{p/2}.
\end{aligned}$$

Under these assumptions, the mappings $f : L^p(\mathcal{I}) \times L^2(\mathcal{I}) \to L^{p'}(\mathcal{I})$ and $g : L^p(\mathcal{I}) \times L^2(\mathcal{I}) \to L^2(\mathcal{I})$ are well defined and satisfy (see [8, Lemma 25]):

$$\begin{aligned} \|f(u,w)\|_{L^{p'}(\mathcal{I})} &\leq A_1|\mathcal{I}|^{1/p'} + A_2\|u\|^{p/p'}_{L^p(\mathcal{I})} + A_3\|w\|^{2/p'}_{L^2(\mathcal{I})}, \\ \|g(u,w)\|_{L^2(\mathcal{I})} &\leq B_1|\mathcal{I}|^{\frac{1}{2}} + B_2\|u\|^{p/2}_{L^p(\mathcal{I})} + B_3\|w\|_{L^2(\mathcal{I})}. \end{aligned} \tag{2.3}$$

In inequalities (2.3) the terms A_1, A_2, A_3, and B_1, B_2, B_3 stand for positive constants that depend on the parameters p and c_j for $j = 1 \dots 6$.

Assumption A_3 (minoration)**.** *The functions f and g satisfy the minoration (2.4)(see [24]). There exist $C_0 \in \mathbb{R}$, $C_1 > 0$, $C_2 > 0$ and $r > 0$ such that for any $(u,w) \in \mathbb{R}^2$,*

$$C_0 + C_1|u|^p + C_2|w|^2 \leq \frac{r}{C} f(u,w)u - g(u,w)w, \tag{2.4}$$

with C the cellular membrane capacitance.

Further, from (2.4), we also derive the minoration below

$$-|C_0| - (r|u|^2 + |w|^2) + C_1|u|^p \leq \frac{r}{C} f(u,w)u - g(u,w)w. \tag{2.5}$$

Let us give the definition of a global weak $T-$ periodic solution.

Definition 1. *Let $T > 0$ and let $\mathcal{I}_T = (0,T) \times \mathcal{I}$. The couple (u,w) is a global weak $T-$ periodic solution of (2.1) if and only if for all $T > 0$, the solution (u,w) satisfies*

$$u \in L^p(\mathcal{I}_T) \cap L^2(0,T;V), \quad w \in L^2(\mathcal{I}_T)$$

and for all $T-$ periodic test functions $\varphi \in \mathscr{C}^1([0,T], H^1(\mathcal{I}))$ and $\xi \in \mathscr{C}^1([0,T], L^2(\mathcal{I}))$ satisfying $\varphi(0,\cdot) = \varphi(T,\cdot)$ and $\xi(0,\cdot) = \xi(T,\cdot)$, we have

$$\begin{aligned} &- C \iint_{\mathcal{I}_T} u(t,x)\partial_t\varphi(t,x)\,dx\,dt + \iint_{\mathcal{I}_T} f(u(t,x),w(t,x))\varphi(t,x)\,dx\,dt \\ &\quad + \iint_{\mathcal{I}_T} \sigma(x)\partial_x u(t,x)\partial_x\varphi(t,x)\,dx\,dt = \int_0^T s(t)\varphi(t,L)\,dt, \\ &- \iint_{\mathcal{I}_T} w(t,x)\partial_t\xi(t,x)\,dx\,dt = \iint_{\mathcal{I}_T} g(u(t,x),w(t,x))\xi(t,x)\,dx\,dt, \end{aligned} \tag{2.6a}$$

with

$$u(0,x) = u(T,x), \qquad w(0,x) = w(T,x), \quad \forall x \in \mathcal{I}. \tag{2.6b}$$

The existence of such weak periodic solution in the sense of Definition 1 has already been established (see for instance [31]). We also mention other proofs in a different spirit, such as [23] for the monodomain formulation and [24] for the bidomain formulation. In this work, we focus on proving the existence of such a solution in the discrete case and its convergence towards the weak solution.

3 Discrete Problem via the finite volume method

We propose a discretization of (2.1) based on the backward Euler implicit scheme in time and a cell-centered finite volume method in space. We then establish its discrete variational formulation (3.9), and prove the existence and uniqueness of the corresponding discrete solution.

3.0.1 Time discretization

Let $T > 0$ denote the period. For the time discretization, we introduce a division of the interval $[0,T]$ into N_T sub-intervals $(t_n, t_{n+1}]$, for $n \in [\![0, N_T - 1]\!]$, such that

$$0 = t_0 < t_1 < \cdots < t_{N_T} = T,$$

with $t_n = n\delta t$, and δt is the time step.

3.0.2 Space discretization

We denote by $\mathcal{I} := (0, L)$ the spatial interval of length $L > 0$. We divide $\mathcal{I}$ into M control volumes K_i, each one defined by the interval $(x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}})$, with edge points $x_{i+\frac{1}{2}}$ for $i = 0, \ldots, M$. The cell centers are denoted by x_i for all $i = 1, \cdots, M$. We denote by $h_{i+\frac{1}{2}}$ for the measure of the interval formed by two consecutive cell centers x_i and x_{i+1}, with $h_{i+\frac{1}{2}} := x_{i+1} - x_{i-1}$ for all $i = 1, \ldots, M-1$. Additionally, we set for the boundaries $x_{\frac{1}{2}}$ and $x_{M+\frac{1}{2}}$ the values $h_{\frac{1}{2}} := x_1 - x_{\frac{1}{2}}$ and $h_{M+\frac{1}{2}} := x_{M+\frac{1}{2}} - x_M$.

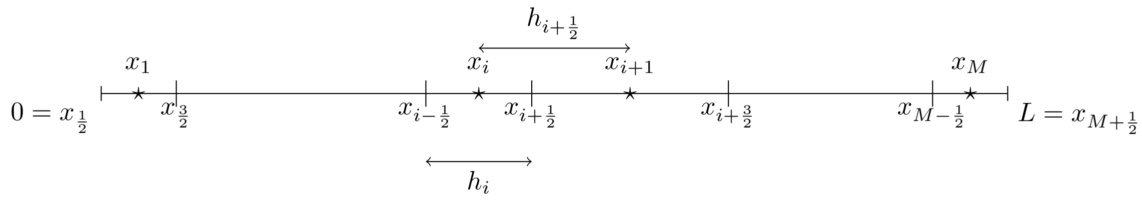

Figure 1: 1D mesh

3.0.3 Space-time discretization

We write $\mathcal{I}_T$ for the space-time domain, such that $\overline{\mathcal{I}_T} := [0, T] \times [0, L]$. First, we consider an admissible space discretization $\mathcal{T}_h$ of $\mathcal{I}$ verifying

$$\mathcal{T}_h := \left((K_i)_{i=1,\ldots,M}, (x_i)_{i=1,\ldots,M}, (x_{i+\frac{1}{2}})_{i=0,\ldots,M}\right),$$

with $x_{i-\frac{1}{2}}$ and $x_{i+\frac{1}{2}}$ the bounds of the cells K_i.
Hence, we introduce the admissible discretization $\mathcal{D}(\mathcal{I}_T)$ of the space-time domain $\mathcal{I}_T$ as

$$\mathcal{D}(\mathcal{I}_T) := \left((t_n)_{0 \leq n \leq N_T}, \mathcal{T}_h\right),$$

for a certain maximum value of space steps h defined by $h = \max_i(h_i, h_{i+\frac{1}{2}})$.

We define a dual partition $\mathcal{D}_h$ such that $[0, L] = \bigcup_{D \in \mathcal{D}_h} \bar{D}$. We denote $D_{i+\frac{1}{2}} = (x_i, x_{i+1})$ for the dual element associated with each node $x_{i+\frac{1}{2}}$, for all $i \in [\![1, M-1]\!]$. We also set $D_{\frac{1}{2}} = (0, x_1)$ and $D_{M+\frac{1}{2}} = (x_M, L)$ for the boundary dual elements respectively associated with $x_{\frac{1}{2}}$ and $x_{M+\frac{1}{2}}$.

3.0.4 Discrete functions

Definition 2. *We introduce the following set L^2_h defined by*

$$L^2_h := \{u_h \in L^2(\mathcal{I}) \ s.t. \ u_h \,|_{K_i} = u_i, \quad \forall i \in [\![1, M]\!]\}$$

such that a function $u_h \in L^2_h$ is associated with a vector $(u_i)_{1 \leq i \leq M}$.

For two given functions u_h, w_h in L^2_h, the $L^2(\mathcal{I})$ scalar product and the $L^2(\mathcal{I})$- norm are

$$(u_h, w_h) = \sum_{i=1}^{M} h_i u_i w_i, \quad \|u_h\|_{L^2(\mathcal{I})} = \left(\sum_{i=1}^{M} h_i |u_i|^2\right)^{\frac{1}{2}}.$$

Let $\mathcal{H}^1_h \subset L^2_h$ denote the subset associated with the discrete norm $\|\cdot\|_{\mathcal{H}^1_h}$ defined as follows:

$$\|u_h\|_{\mathcal{H}^1_h} := \sqrt{\|u_h\|^2_{L^2(\mathcal{I})} + \|\partial^h_x u_h\|^2_{L^2(\mathcal{I})}}. \tag{3.1}$$

Definition 3. *We define ∂^h_x as the discrete gradient of u_h in L^2_h. The operator ∂^h_x is expressed as follows:*

$$\partial^h_x u_h := \begin{cases} \frac{u_1 - u_0}{h_{\frac{1}{2}}}, \ \text{if } x \in D_{\frac{1}{2}}, \\ \frac{u_{i+1} - u_i}{h_{i+\frac{1}{2}}}, \ \text{if } x \in D_{i+\frac{1}{2}}, i = 1, \cdots, M-1, \\ \frac{u_{M+1} - u_M}{h_{M+\frac{1}{2}}}, \ \text{if } x \in D_{M+\frac{1}{2}}. \end{cases} \tag{3.2}$$

Therefrom, we have

$$(\partial_x^h u_h, \partial_x^h w_h) = \sum_{i=0}^{M} h_{i+\frac{1}{2}} \left(\frac{u_{i+1}-u_i}{h_{i+\frac{1}{2}}}\right)\left(\frac{w_{i+1}-w_i}{h_{i+\frac{1}{2}}}\right),$$

$$\|\partial_x^h u_h\|_{L^2(\mathcal{I})} = \left(\sum_{i=0}^{M} h_{i+\frac{1}{2}} \left|\frac{u_{i+1}-u_i}{h_{i+\frac{1}{2}}}\right|^2\right)^{\frac{1}{2}}.$$

3.1 Discrete form of the monodomain model

In this part, we introduce the finite volume scheme (3.5a)-(3.5d) and the discrete variational formulation (3.9) associated to (2.1)-(2.2). We prove at each time step $n \in [\![0, N_T - 1]\!]$, the existence and uniqueness of the discrete solution $(u_h^{n+1}, w_h^{n+1})_{h>0,n\geq 0}$. We remark that the uniqueness is ensured under a condition on the time step δt.

3.1.1 The cell-centered finite volume method

Now, we discretize the system of equations (2.1)-(2.2) using the cell-centered finite volume method. It consists in approximating the value of (u^n, w^n) at each time t^n, $\forall n \in [\![1, N_T]\!]$, using one value (u_i^n, w_i^n) per cell K_i. The resulting vector $(\boldsymbol{U}_h^n, \boldsymbol{W}_h^n) \in \mathbb{R}^{M+2} \times \mathbb{R}^M$ verifies

$$[\boldsymbol{U}_h^n]_i := u_i^n, \quad \forall i \in [\![0, M+1]\!], \quad [\boldsymbol{W}_h^n]_i := w_i^n, \quad \forall i \in [\![1, M]\!],$$

with $[\boldsymbol{U}_h^n]_0 := u^n(x_{\frac{1}{2}})$ and $[\boldsymbol{U}_h^n]_{M+1} := u^n(x_{M+\frac{1}{2}})$, being the discrete values of u^n at the boundaries of $\mathcal{I}$.

We integrate the system (2.1)-(2.2) over $(t_n, t_{n+1}) \times K_i$ for all $n \in [\![0, N_T - 1]\!]$ and $i \in [\![1, M]\!]$ to get

$$\begin{aligned} &C\int_{K_i} (u(t_{n+1}, x) - u(t_n, x))\, dx - \int_{t_n}^{t_{n+1}} \Big(\sigma(x_{i+\frac{1}{2}})\partial_x u(t, x_{i+\frac{1}{2}}) - \sigma(x_{i-\frac{1}{2}})\partial_x u(t, x_{i-\frac{1}{2}})\Big)\, dt \\ &\qquad + \int_{t_n}^{t_{n+1}} \int_{K_i} f(u(t,x), w(t,x))\, dt\, dx = 0, \\ &\int_{K_i} (w(t_{n+1}, x) - w(t_n, x))dx = \int_{t_n}^{t_{n+1}} \int_{K_i} g(u(t,x), w(t,x))\, dt\, dx. \end{aligned} \tag{3.3}$$

We consider the following approximations

$$\begin{aligned} u_i^{n+1} &\approx \frac{1}{h_i}\int_{K_i} u(t_{n+1}, x)\ dx, \qquad w_i^{n+1} \approx \frac{1}{h_i}\int_{K_i} w(t_{n+1}, x)\ dx, \\ f(u_i^{n+1}, w_i^{n+1}) &\approx \frac{1}{\delta t h_i}\int_{t_n}^{t_{n+1}} \int_{K_i} f(u(t_{n+1}, x), w(t_{n+1}, x))\, dx\, dt, \\ g(u_i^{n+1}, w_i^{n+1}) &\approx \frac{1}{\delta t h_i}\int_{t_n}^{t_{n+1}} \int_{K_i} g(u(t_{n+1}, x), w(t_{n+1}, x))\, dx\, dt. \end{aligned}$$

Additionally, the flux across each edge $x_{i+\frac{1}{2}}$ is approximated by

$$\sigma(x_{i+\frac{1}{2}})\partial_x u(t_{n+1}, x_{i+\frac{1}{2}}) \approx F_{i+\frac{1}{2}}^{n+1} := \begin{cases} \tau_{\frac{1}{2}}(u_1^{n+1} - u_0^{n+1}),\ i = 0 \\ \tau_{i+\frac{1}{2}}(u_{i+1}^{n+1} - u_i^{n+1}),\ i \in [\![1, M-1]\!], \\ \tau_{M+\frac{1}{2}}(u_{M+1}^{n+1} - u_M^{n+1}),\ i = M, \end{cases} \tag{3.4}$$

with $\sigma_{i+\frac{1}{2}} := \sigma(x_{i+\frac{1}{2}})$ and $\tau_{i+\frac{1}{2}} = \frac{\sigma_{i+\frac{1}{2}}}{h_{i+\frac{1}{2}}}$.

Thus, combining (3.3) and (3.4) and considering the Neumann boundary conditions introduced in (2.1), we get at each time step $n \in [\![0, N_T - 1]\!]$ the following system of nonlinear algebraic equations

$$F_{\frac{1}{2}}^{n+1} = 0, \tag{3.5a}$$

$$Ch_i(u_i^{n+1}-u_i^n)-\delta t(F_{i+\frac{1}{2}}^{n+1}-F_{i-\frac{1}{2}}^{n+1})+\delta t h_i f(u_i^{n+1},w_i^{n+1})=0,\quad \forall i\in [\![1,M]\!], \tag{3.5b}$$

$$F_{M+\frac{1}{2}}^{n+1}=s^{n+1}, \tag{3.5c}$$

$$h_i(w_i^{n+1}-w_i^n)=h_i\delta t g(u_i^{n+1},w_i^{n+1}),\quad \forall i\in [\![1,M]\!], \tag{3.5d}$$

with the initial conditions approximated over each control volume K_i by

$$u_i^0:=\frac{1}{h_i}\int_{K_i}u(0,x)\,dx,\quad w_i^0:=\frac{1}{h_i}\int_{K_i}w(0,x)\,dx,\quad \forall i\in [\![1,M]\!]. \tag{3.6}$$

For each $n\in [\![0,N_T-1]\!]$, we introduce the nonlinear function $\mathcal{S}_n : X\in\mathbb{R}^{2M+2}\mapsto S(X)\in\mathbb{R}^{2M+2}$ defined by

$$[\mathcal{S}_n(\boldsymbol{X})]_0 := F_{1/2}^{n+1},$$

$$[\mathcal{S}_n(\boldsymbol{X})]_i := Ch_i(u_i^{n+1}-u_i^n)-\delta t\left(F_{i+1/2}^{n+1}-F_{i-1/2}^{n+1}\right)+\delta t h_i f(u_i^{n+1},w_i^{n+1}),\quad \forall i\in [\![1,M]\!],$$

$$[\mathcal{S}_n(\boldsymbol{X})]_{M+1} := F_{M+1/2}^{n+1},$$

$$[\mathcal{S}_n(\boldsymbol{X})]_i := h_i(w_i^{n+1}-w_i^n)-\delta t h_i g(u_i^{n+1},w_i^{n+1}),\quad \forall i\in [\![M+2,2M+2]\!].$$

The finite volume scheme of (2.1) is equivalent to solving $\mathcal{S}_n(\boldsymbol{X})=\mathbf{0}$ at each time t^{n+1} with $\boldsymbol{X}$ the vector of all the discrete unknowns.

The initial conditions are approximated over each control volume K_i by

$$u_i^0:=\frac{1}{h_i}\int_{K_i}u(0,x)\,dx,\quad w_i^0:=\frac{1}{h_i}\int_{K_i}w(0,x)\,dx,\quad \forall i\in [\![1,M]\!]. \tag{3.8}$$

3.1.2 Discrete Variational Formulation

The discrete problem (3.5a)-(3.5d) can also be written under a discrete variational form.

Lemma 1. *Let $n\in [\![0,N_T-1]\!]$ be a time step. We consider the finite volume scheme (3.5a)-(3.5d). Then, it is equivalent to the following discrete variational formulation:*

$$\begin{aligned} C\int_{\mathcal{I}}(u_h^{n+1}(x)-u_h^n(x))v_h(x)\,dx+\delta t\int_{\mathcal{I}}\sigma_h(x)\partial_x^h u_h^{n+1}(x)\partial_x^h v_h(x)\,dx\\ +\delta t\int_{\mathcal{I}}f(u_h^{n+1}(x),w_h^{n+1}(x))v_h(x)\,dx=\delta t s^{n+1}v_{M+1},\quad \forall v_h\in\mathcal{H}_h^1,\end{aligned} \tag{3.9a}$$

$$\int_{\mathcal{I}}(w_h^{n+1}(x)-w_h^n(x))z_h(x)\,dx=\delta t\int_{\mathcal{I}}g(u_h^{n+1}(x),w_h^{n+1}(x))z_h(x)\,dx,\quad \forall z_h\in L_h^2, \tag{3.9b}$$

where the functions $u_h^{n+1},w_h^{n+1},f(u_h^{n+1},w_h^{n+1}),g(u_h^{n+1},w_h^{n+1})$ are constant per control volume, and the function σ_h is constant per diamond. It satisfies

$$\begin{aligned}&u_h^{n+1}|_{K_i}:=u_i^{n+1},\quad w_h^{n+1}|_{K_i}:=w_i^{n+1},\quad f(u_h^{n+1},w_h^{n+1})|_{K_i}:=f(u_i^{n+1},w_i^{n+1}),\\ &g(u_h^{n+1},w_h^{n+1})|_{K_i}:=g(u_i^{n+1},w_i^{n+1}),\ \textit{for all } i\in [\![1,M]\!],\\ &\sigma_h|_{D_{i+\frac{1}{2}}}:=\sigma_{i+\frac{1}{2}},\ \textit{for all } i\in [\![0,M]\!].\end{aligned}$$

3.1.3 Existence and uniqueness of a discrete solution $(u_h^{n+1},w_h^{n+1})_{h>0,n\geq 0}$

Let $h>0$ and $n\in [\![0,N_T-1]\!]$. Given (u_h^n,w_h^n), we establish the existence of a solution over the interval $(t_n,t_{n+1}]$ of the discrete variational system (3.9) using a consequence of Brouwer's theorem for the fixed points (see [33]). The solution (u_h^{n+1},w_h^{n+1}) is even unique under a suitable assumption on δt.

First, we recall the following Lemma provided in [33, 5, 6]

Lemma 2. *Let $\mathcal{H}$ be a finite-dimensional Hilbert space, equipped with a scalar product $\langle \cdot, \cdot \rangle$. We define the function $\mathcal{P}$ such that $\mathcal{P} : (\mathcal{H}, \|\cdot\|_{\mathcal{H}}) \to (\mathcal{H}, \|\cdot\|_{\mathcal{H}})$. Assume $\exists \alpha > 0$ such that*

$$\langle \mathcal{P}(\xi), \xi \rangle > 0, \text{ for } \|\xi\|_{\mathcal{H}} = \alpha > 0$$

Then,

$$\exists \xi^* \in \mathcal{H}, \quad \text{with } \|\xi^*\|_{\mathcal{H}} \leq \alpha \text{ s.t. } \mathcal{P}(\xi^*) = 0.$$

Further, we recall the trace inequality.

Proposition 1 (Trace inequality)**.** *For any function $u_h \in L^2_h$ constant over each control volume of the mesh $\mathcal{T}_h$, and associated with a vector $(u_i)_{0 \leq i \leq M+1}$, the following inequality holds*

$$\left| u_{M+1} - \frac{1}{L} \sum_{i=1}^{M} h_i u_i \right| \leq L \|\partial_x^h u_h\|_{L^2(\mathcal{I})}. \tag{3.10}$$

Consequently, by using the Cauchy-Schwarz inequality, we get

$$|u_{M+1}| \leq L \|\partial_x^h u_h\|_{L^2(\mathcal{I})} + \frac{1}{\sqrt{L}} \|u_h\|_{L^2(\mathcal{I})}. \tag{3.11}$$

Proof.

$$\left| u_{M+1} - \frac{1}{L} \sum_{i=1}^{M} h_i u_i \right| = \frac{1}{L} \left| L u_{M+1} - \sum_{i=1}^{M} h_i u_i \right| = \frac{1}{L} \left| \sum_{i=1}^{M} h_i (u_{M+1} - u_i) \right|.$$

Observe that $|u_{M+1} - u_i| \leq L \|\partial_x^h u_h\|_{L^2(\mathcal{I})}$, so we deduce

$$\left| u_{M+1} - \frac{1}{L} \sum_{i=1}^{M} h_i u_i \right| \leq \sum_{i=1}^{M} h_i \|\partial_x^h u_h\|_{L^2(\mathcal{I})} = \|\partial_x^h u_h\|_{L^2(\mathcal{I})} \sum_{i=1}^{M} h_i = L \|\partial_x^h u_h\|_{L^2(\mathcal{I})}.$$

□

In the sequel, we consider the Hilbert finite-dimensional space $\mathcal{H}^1_h \times L^2_h$ equipped with the norm $\|\cdot\|_{\mathcal{H}^1_h \times L^2_h}$ expressed by

$$\|(u_h, w_h)\|_{\mathcal{H}^1_h \times L^2_h} := \sqrt{r \|u_h\|^2_{\mathcal{H}^1_h} + \|w_h\|^2_{L^2_h}},$$

where $r > 0$ is given by Assumption A_3.

Now, we introduce the following Lemma.

Lemma 3. *Let $\mathcal{P}$ be a nonlinear continuous operator from the finite-dimensional Hilbert space $\mathcal{H}^1_h \times L^2_h$ into itself; it satisfies $\forall (v_h, z_h) \in \mathcal{H}^1_h \times L^2_h$*

$$\begin{aligned} \langle \mathcal{P}(u_h^{n+1}, w_h^{n+1}), (v_h, z_h) \rangle &= r \int_{\mathcal{I}} (u_h^{n+1}(x) - u_h^n(x)) v_h(x) \; dx \\ &+ \int_{\mathcal{I}} (w_h^{n+1}(x) - w_h^n(x)) z_h(x) \; dx + \frac{r\delta t}{C} \int_{\mathcal{I}} \sigma_h(x) \partial_x^h u_h^{n+1}(x) \partial_x^h v_h^{n+1}(x) \; dx \\ &- \frac{r\delta t}{C} s^{n+1} v_{M+1} + \delta t \int_{\mathcal{I}} \left(\frac{r}{C} f(u_h^{n+1}(x), w_h^{n+1}(x)) v_h(x) - g(u_h^{n+1}(x), w_h^{n+1}(x)) z_h(x) \right) \; dx. \end{aligned} \tag{3.12}$$

Further, the function $\mathcal{P}$ satisfies

$$\langle \mathcal{P}(u_h^{n+1}, w_h^{n+1}), (u_h^{n+1}, w_h^{n+1}) \rangle > 0, \tag{3.13}$$

for $\|(u_h^{n+1}, w_h^{n+1})\|_{\mathcal{H}^1_h \times L^2_h} = R(n)$, with $R(n)$ a positive constant depending on the time iteration n.

For $0 < \delta t < \frac{1}{4}$, this constant is given by

$$R^2(n) := \frac{2}{k_1(\delta t)} \left(\frac{1}{2} \left(r \|u_h^n\|^2_{L^2(\mathcal{I})} + \|w_h^n\|^2_{L^2(\mathcal{I})} \right) + \delta t k_2 \right), \tag{3.14}$$

with

$$k_1(\delta t) = \frac{1}{2}\min\left(1-4\delta t, \frac{\delta t\sigma_*}{C}\right), \qquad k_2 = \frac{r}{2}\left(\frac{1}{2LC^2}+\frac{L^2}{C\sigma_*}\right)\|s\|^2_{L^\infty(0,T)} + |C_0|L > 0,$$

where C_0 and σ_* denote the constants from Assumptions A_3 and A_1, respectively.

Proof. Let $r>0$, $h>0$ and $n \in [\![0, N_T-1]\!]$. We set $(v_h, z_h) = (u_h^{n+1}, w_h^{n+1})$, with

$$\|(u_h^{n+1}, w_h^{n+1})\|_{\mathcal{H}_h^1\times L_h^2} = R(n), \tag{3.15}$$

Thus, the inner product of $\mathcal{P}(u_h^{n+1}, w_h^{n+1})$ and (v_h, z_h) becomes

$$\begin{aligned}
&\langle \mathcal{P}(u_h^{n+1}, w_h^{n+1}), (u_h^{n+1}, w_h^{n+1})\rangle = r\int_{\mathcal{I}} (u_h^{n+1}(x) - u_h^n(x))u_h^{n+1}(x)\,dx \\
&+ \int_{\mathcal{I}} (w_h^{n+1}(x) - w_h^n(x))w_h^{n+1}(x)\,dx + \frac{r\delta t}{C}\int_{\mathcal{I}} \sigma_h(x)\partial_x^h u_h^{n+1}(x)\partial_x^h u_h^{n+1}(x)\,dx \\
&+ \delta t\int_{\mathcal{I}}\left(\frac{r}{C}f(u_h^{n+1}(x), w_h^{n+1}(x))u_h^{n+1}(x) - g(u_h^{n+1}(x), w_h^{n+1}(x))w_h^{n+1}(x)\right)\,dx \\
&- \frac{r\delta t}{C}s^{n+1}u_{M+1}^{n+1}.
\end{aligned}$$

As an intermediate step, we introduce the following results.

First, we apply the inequality (2.5) in Assumption A_3 to (u_h^{n+1}, w_h^{n+1}) to obtain

$$\begin{aligned}
&\int_{\mathcal{I}} \frac{r}{C}f(u_h^{n+1}(x), w_h^{n+1}(x))u_h^{n+1}(x)\,dx - g(u_h^{n+1}(x), w_h^{n+1}(x))w_h^{n+1}(x)\,dx \\
&\quad\geq C_1\|u_h^{n+1}\|^p_{L^p(\mathcal{I})} - \left(r\|u_h^{n+1}\|^2_{L^2(\mathcal{I})} + \|w_h^{n+1}\|^2_{L^2(\mathcal{I})}\right) - |C_0|L.
\end{aligned} \tag{3.16}$$

Further, let s be the $T-$ periodic function. Using trace (3.11) and Young's inequalities, we deduce that

$$\frac{r\delta t}{C}|s^{n+1}u_{M+1}| \leq \frac{r\delta t}{2}\left(\frac{L^2}{C\theta}+\frac{1}{LC^2\epsilon}\right)\|s\|^2_{L^\infty(0,T)} + \frac{r\delta t\theta}{2C}\|\partial_x^h u_h^{n+1}\|^2_{L^2(\mathcal{I})} + \frac{r\delta t\epsilon}{2}\|u_h^{n+1}\|^2_{L^2(\mathcal{I})}, \tag{3.17}$$

for all $\epsilon > 0$ and $\theta > 0$.

Indeed, for any $\epsilon > 0$ and $\theta > 0$, we have

$$\begin{aligned}
\frac{r\delta t}{C}|s^{n+1}u_{M+1}| &\leq \frac{r\delta t}{C}|s^{n+1}|L\|\partial_x^h u_h^{n+1}\|_{L^2(\mathcal{I})} + \frac{r\delta t}{C}|s^{n+1}|\frac{1}{\sqrt{L}}\|u_h^{n+1}\|_{L^2(\mathcal{I})} \\
&\leq \frac{r\delta t}{2}\left(\frac{L^2}{C\theta}+\frac{1}{LC^2\epsilon}\right)\|s\|^2_{L^\infty(0,T)} + \frac{r\delta t\theta}{2C}\|\partial_x^h u_h^{n+1}\|^2_{L^2(\mathcal{I})} + \frac{r\delta t\epsilon}{2}\|u_h^{n+1}\|^2_{L^2(\mathcal{I})}.
\end{aligned}$$

Returning to the main proof, we now use the inequality

$$(a-b)a \geq \tfrac{1}{2}(a^2-b^2)$$

on the first two terms, together with the minoration σ_* of the function σ (see Assumption A_1), as well as inequality (3.16) and inequality (3.17), with the choices $\theta = \sigma_*$ and $\epsilon = 2$ to get the following estimate:

$$\begin{aligned}
&\langle \mathcal{P}(u_h^{n+1}, w_h^{n+1}), (u_h^{n+1}, w_h^{n+1})\rangle \geq \frac{1}{2}\left(r\|u_h^{n+1}\|^2_{L^2(\mathcal{I})} + \|w_h^{n+1}\|^2_{L^2(\mathcal{I})}\right) - \frac{1}{2}\left(r\|u_h^{n}\|^2_{L^2(\mathcal{I})} + \|w_h^{n}\|^2_{L^2(\mathcal{I})}\right) \\
&\quad \frac{r\delta t\sigma_*}{2C}\|\partial_x^h u_h^{n+1}\|^2_{L^2(\mathcal{I})} + C_1\delta t\|u_h^{n+1}\|^p_{L^p(\mathcal{I})} - |C_0|\delta tL - \delta t\left(r\|u_h^{n+1}\|^2_{L^2(\mathcal{I})} + \|w_h^{n+1}\|^2_{L^2(\mathcal{I})}\right) \\
&\quad - \frac{r\delta t}{2}\left(\frac{1}{2LC^2}+\frac{L^2}{C\sigma_*}\right)\|s\|^2_{L^\infty(0,T)} - r\delta t\|u_h^{n+1}\|^2_{L^2(\mathcal{I})}.
\end{aligned}$$

Then, we obtain

$$\langle \mathcal{P}(u_h^{n+1}, w_h^{n+1}), (u_h^{n+1}, w_h^{n+1})\rangle \geq \left(\frac{1}{2} - 2\delta t\right)\left(r\|u_h^{n+1}\|^2_{L^2(\mathcal{I})} + \|w_h^{n+1}\|^2_{L^2(\mathcal{I})}\right)$$
$$- \frac{1}{2}\left(r\|u_h^n\|^2_{L^2(\mathcal{I})} + \|w_h^n\|^2_{L^2(\mathcal{I})}\right) + \frac{r\delta t\sigma_*}{2C}\|\partial_x^h u_h^{n+1}\|^2_{L^2(\mathcal{I})} + C_1\delta t\|u_h^{n+1}\|^p_{L^p(\mathcal{I})}$$
$$- \delta t\left(\frac{r}{2}\left(\frac{1}{2LC^2} + \frac{L^2}{C\sigma_*}\right)\|s\|^2_{L^\infty(0,T)} + |C_0|L\right).$$

We set

$$k_1(\delta t) = \min\left(\frac{1}{2} - 2\delta t, \frac{\delta t\sigma_*}{2C}\right), \quad k_2 = \frac{r}{2}\left(\frac{1}{2LC^2} + \frac{L^2}{C\sigma_*}\right)\|s\|^2_{L^\infty(0,T)} + |C_0|L > 0,$$

with $0 < \delta t < \frac{1}{4}$.

As a result, we have

$$\langle \mathcal{P}(u_h^{n+1}, w_h^{n+1}), (u_h^{n+1}, w_h^{n+1})\rangle \geq k_1(\delta t)\left(r\left(\|u_h^{n+1}\|^2_{L^2(\mathcal{I})} + \|\partial_x^h u_h^{n+1}\|^2_{L^2(\mathcal{I})}\right) + \|w_h^{n+1}\|^2_{L^2(\mathcal{I})}\right)$$
$$- \frac{1}{2}\left(r\|u_h^n\|^2_{L^2(\mathcal{I})} + \|w_h^n\|^2_{L^2(\mathcal{I})}\right) - \delta t k_2.$$

In other words, we deduce that

$$\langle \mathcal{P}(u_h^{n+1}, w_h^{n+1}), (u_h^{n+1}, w_h^{n+1})\rangle \geq k_1(\delta t)\|(u_h^{n+1}, w_h^{n+1})\|^2_{\mathcal{H}_h^1\times L_h^2} - \frac{1}{2}\left(r\|u_h^n\|^2_{L^2(\mathcal{I})} + \|w_h^n\|^2_{L^2(\mathcal{I})}\right)$$
$$- \delta t k_2.$$

We deduce that

$$\langle \mathcal{P}(u_h^{n+1}, w_h^{n+1}), (u_h^{n+1}, w_h^{n+1})\rangle \geq \left(k_1(\delta t) - \frac{k_1(\delta t)}{2}\right)R^2(n) = \frac{k_1(\delta t)R^2(n)}{2} > 0.$$

□

Theorem 1. *Let $h > 0$ and $n \in [\![0, N_T - 1]\!]$. We consider $r > 0$ as in Assumption A_3. We introduce the function $\mathcal{P}$ defined in (3.12) and the radius $R(n)$ given in (3.14) (see Lemma 3). Then, for any given data (u_h^n, w_h^n), the discrete variational system (3.9) admits a solution still denoted (u_h^{n+1}, w_h^{n+1}) in the ball $\bar{B}(0, R(n)) \subset \mathcal{H}_h^1 \times L_h^2$, for $0 < \delta t < \frac{1}{4}$.*

Additionally, this solution is unique under the condition

$$0 < \delta t\left(\frac{r^{1/2}}{C}L_f L^{1/2} + \frac{\sigma^*}{C} + L_g L^{1/2}\right) < 1. \tag{3.18}$$

Proof. Let $h > 0$ and $n \in [\![0, N_T - 1]\!]$.

From Lemma 3, we have

$$\langle \mathcal{P}(u_h^{n+1}, w_h^{n+1}), (u_h^{n+1}, w_h^{n+1})\rangle > 0,$$

for $\|(u_h^{n+1}, w_h^{n+1})\|_{\mathcal{H}_h^1\times L_h^2} = R(n)$, with $R(n)$ defined in (3.14).

Then, by Lemma 2, there exists a solution $(u_h^{n+1}, w_h^{n+1}) \in \overline{B}\big(0, R(n)\big)$ included in $\mathcal{H}_h^1 \times L_h^2$ such that

$$\mathcal{P}(u_h^{n+1}, w_h^{n+1}) = \mathbf{0}.$$

Then, we proved the existence of a solution (u_h^{n+1}, w_h^{n+1}) to the discrete problem (3.9).

In what follows, we demonstrate the uniqueness of the solution (u_h^{n+1}, w_h^{n+1}) under a suitable time step δt.

Let us introduce the nonlinear continuous operator Q defined on the subset $\bar{B}(0,R(n))$ of $\mathcal{H}_h^1 \times L_h^2$ into $\mathcal{H}_h^1 \times L_h^2$ satisfying

$$\begin{aligned}\langle Q(u_h^{n+1},w_h^{n+1}),(v_h,z_h)\rangle &= r\int_{\mathcal{I}} u_h^n(x)v_h(x)\ dx + \int_{\mathcal{I}} w_h^n(x)z_h(x)\ dx\\ &- \frac{r\delta t}{C}\int_{\mathcal{I}}\sigma_h(x)\partial_x^h u_h^{n+1}(x)\partial_x^h v_h(x)\ dx - \frac{r\delta t}{C}\int_{\mathcal{I}} f(u_h^{n+1}(x),w_h^{n+1}(x))v_h(x)\ dx\\ &+\ \delta t\int_{\mathcal{I}} g(u_h^{n+1}(x),w_h^{n+1}(x))z_h(x)\ dx + \frac{r\delta t}{C}s^{n+1}v_{M+1},\end{aligned}$$

for all $(u_h^{n+1},w_h^{n+1}) \in \bar{B}(0,R(n))$ and $(v_h,w_h)\in\mathcal{H}_h^1\times L_h^2$.
Therefore, the following statements are equivalent:

$$\mathcal{P}(u_h^{n+1},w_h^{n+1}) = \mathbf{0} \Leftrightarrow Q(u_h^{n+1},w_h^{n+1}) = (u_h^{n+1},w_h^{n+1}).$$

In other words, (u_h^{n+1},w_h^{n+1}) is a solution to $\mathcal{P}$ if and only if it is also a fixed point to Q.
The solution (u_h^{n+1},w_h^{n+1}) is unique. Indeed, let (u_h^{n+1},w_h^{n+1}) and $(\widehat{u}_h^{n+1},\widehat{w}_h^{n+1})$ belonging to $\bar{B}(0,R(n))$ be two distinct solutions to the discrete variational system (3.9) satisfying

$$Q(u_h^{n+1},w_h^{n+1}) = (u_h^{n+1},w_h^{n+1}) \quad \text{and} \quad Q(\widehat{u}_h^{n+1},\widehat{w}_h^{n+1}) = (\widehat{u}_h^{n+1},\widehat{w}_h^{n+1}). \tag{3.19}$$

We note that $Q(u_h^{n+1},w_h^{n+1}) - Q(\widehat{u}_h^{n+1},\widehat{w}_h^{n+1}) \in \mathcal{H}_h^1\times L_h^2$ and its norm can be expressed using the dual norm of $\mathcal{H}_h^1\times L_h^2$ with

$$\begin{aligned}&\|Q(u_h^{n+1},w_h^{n+1}) - Q(\widehat{u}_h^{n+1},\widehat{w}_h^{n+1})\|_{\mathcal{H}_h^1\times L_h^2}\\ &= \sup_{\|(v_h,z_h)\|_{\mathcal{H}_h^1\times L_h^2}\le 1} |\langle Q(u_h^{n+1},w_h^{n+1}) - Q(\widehat{u}_h^{n+1},\widehat{w}_h^{n+1}),(v_h,z_h)\rangle|.\end{aligned} \tag{3.20}$$

Now, let $(v_h,z_h)\in\mathcal{H}_h^1\times L_h^2$, with $\|(v_h,z_h)\|_{\mathcal{H}_h^1\times L_h^2} \ \le\ 1$, we have

$$\begin{aligned}\big|\langle Q(u_h^{n+1},w_h^{n+1}) - Q(\widehat{u}_h^{n+1},\widehat{w}_h^{n+1}),(v_h,z_h)\rangle\big| &= \Big| r\int_{\mathcal{I}} \left(u_h^n(x) - u_h^n(x)\right)v_h(x)\ dx\\ &+ \int_{\mathcal{I}}\left(w_h^n(x) - w_h^n(x)\right)z_h(x)\ dx\\ &- \frac{r\delta t}{C}\int_{\mathcal{I}}\left(f\left(u_h^{n+1}(x),w_h^{n+1}(x)\right) - f\left(\widehat{u}_h^{n+1}(x),\widehat{w}_h^{n+1}(x)\right)\right)v_h(x)\ dx\\ &- \frac{r\delta t}{C}\int_{\mathcal{I}}\sigma_h(x)\partial_x^h\left(u_h^{n+1}(x) - \widehat{u}_h^{n+1}(x)\right)\partial_x^h v_h(x)\ dx\\ &+ \delta t\int_{\mathcal{I}}\left(g\left(u_h^{n+1}(x),w_h^{n+1}(x)\right) - g\left(\widehat{u}_h^{n+1}(x),\widehat{w}_h^{n+1}(x)\right)\right)z_h(x)\ dx\Big|.\end{aligned}$$

Thus, we derive

$$\begin{aligned}&|\langle Q(u_h^{n+1},w_h^{n+1}) - Q(\widehat{u}_h^{n+1},\widehat{w}_h^{n+1}),(v_h,z_h)\rangle|\\ &\le\ \frac{r\delta t}{C}\left|\int_{\mathcal{I}}\left(f\left(u_h^{n+1}(x),w_h^{n+1}(x)\right) - f\left(\widehat{u}_h^{n+1}(x),\widehat{w}_h^{n+1}(x)\right)\right)v_h(x)\ dx\right|\\ &+ \frac{r\delta t}{C}\left|\int_{\mathcal{I}}\sigma_h(x)\partial_x^h\left(u_h^{n+1}(x) - \widehat{u}_h^{n+1}(x)\right)\partial_x^h v_h(x)\ dx\right|\\ &+ \delta t\left|\int_{\mathcal{I}}\left(g\left(u_h^{n+1}(x),w_h^{n+1}(x)\right) - g\left(\widehat{u}_h^{n+1}(x),\widehat{w}_h^{n+1}(x)\right)\right)z_h(x)\ dx\right|.\end{aligned}$$

Since f,g are polynomial functions on the compact ball $\bar{B}(0,R(n))$, there exist $L_f>0$ and $L_g>0$ such that

$$\begin{aligned}|f(u_h^{n+1},w_h^{n+1}) - f(\widehat{u}_h^{n+1},\widehat{w}_h^{n+1})| &\le L_f\|(u_h^{n+1} - \widehat{u}_h^{n+1},w_h^{n+1} - \widehat{w}_h^{n+1})\|_{\mathcal{H}_h^1\times L_h^2},\\ |g(u_h^{n+1},w_h^{n+1}) - g(\widehat{u}_h^{n+1},\widehat{w}_h^{n+1})| &\le L_g\|(u_h^{n+1} - \widehat{u}_h^{n+1},w_h^{n+1} - \widehat{w}_h^{n+1})\|_{\mathcal{H}_h^1\times L_h^2}.\end{aligned}$$

Consequently, we obtain

$$\begin{aligned}
&|\langle Q(u_h^{n+1}, w_h^{n+1}) - Q(\widehat{u}_h^{n+1}, \widehat{w}_h^{n+1}), (v_h, z_h)\rangle| \\
&\quad \leq \frac{r\delta t}{C} L_f L^{\frac{1}{2}} \|(u_h^{n+1} - \widehat{u}_h^{n+1}, w_h^{n+1} - \widehat{w}_h^{n+1})\|_{\mathcal{H}_h^1 \times L_h^2} \|v_h\|_{L^2(\mathcal{I})} \\
&\quad + \frac{r\delta t\sigma^*}{C} \|\partial_x^h (u_h^{n+1} - \widehat{u}_h^{n+1})\|_{L^2(\mathcal{I})} \|\partial_x^h v_h\|_{L^2(\mathcal{I})} \\
&\quad + \delta t L_g L^{\frac{1}{2}} \|(u_h^{n+1} - \widehat{u}_h^{n+1}, w_h^{n+1} - \widehat{w}_h^{n+1})\|_{\mathcal{H}_h^1 \times L_h^2} \|z_h\|_{L^2(\mathcal{I})},
\end{aligned}$$

Thus, we have

$$\begin{aligned}
&|\langle Q(u_h^{n+1}, w_h^{n+1}) - Q(\widehat{u}_h^{n+1}, \widehat{w}_h^{n+1}), (v_h, z_h)\rangle| \\
&\quad \leq \frac{r^{\frac{1}{2}}\delta t}{C} L_f L^{\frac{1}{2}} \|(u_h^{n+1} - \widehat{u}_h^{n+1}, w_h^{n+1} - \widehat{w}_h^{n+1})\|_{\mathcal{H}_h^1 \times L_h^2} r^{\frac{1}{2}} \|v_h\|_{L^2(\mathcal{I})} \\
&\quad + \frac{\sigma^*\delta t}{C} \|(u_h^{n+1} - \widehat{u}_h^{n+1}, w_h^{n+1} - \widehat{w}_h^{n+1})\|_{\mathcal{H}_h^1 \times L_h^2} r^{\frac{1}{2}} \|\partial_x^h v_h\|_{L^2(\mathcal{I})} \\
&\quad + \delta t L_g L^{\frac{1}{2}} \|(u_h^{n+1} - \widehat{u}_h^{n+1}, w_h^{n+1} - \widehat{w}_h^{n+1})\|_{\mathcal{H}_h^1 \times L_h^2} \|z_h\|_{L^2(\mathcal{I})}.
\end{aligned}$$

Given that $\|(v_h, z_h)\|_{\mathcal{H}_h^1 \times L_h^2} \leq 1$ implies

$$\begin{aligned}
&|\langle Q(u_h^{n+1}, w_h^{n+1}) - Q(\widehat{u}_h^{n+1}, \widehat{w}_h^{n+1}), (v_h, z_h)\rangle| \\
&\quad \leq \|(u_h^{n+1} - \widehat{u}_h^{n+1}, w_h^{n+1} - \widehat{w}_h^{n+1})\|_{\mathcal{H}_h^1 \times L_h^2} \delta t \left(\frac{r^{\frac{1}{2}}}{C} L_f L^{\frac{1}{2}} + \frac{\sigma^*}{C} + L_g L^{\frac{1}{2}} \right).
\end{aligned} \tag{3.21}$$

Consequently, (3.20) and given (3.21) , we deduce that

$$\begin{aligned}
&\|Q(u_h^{n+1}, w_h^{n+1}) - Q(\widehat{u}_h^{n+1}, \widehat{w}_h^{n+1})\|_{\mathcal{H}_h^1 \times L_h^2} \\
&\quad \leq \delta t \left(\frac{r^{\frac{1}{2}}}{C} L_f L^{\frac{1}{2}} + \frac{\sigma^*}{C} + L_g L^{\frac{1}{2}} \right) \|(u_h^{n+1} - \widehat{u}_h^{n+1}, w_h^{n+1} - \widehat{w}_h^{n+1})\|_{\mathcal{H}_h^1 \times L_h^2}.
\end{aligned}$$

Additionally, given the equality (3.19) and the following assumption:

$$0 < \delta t \left(\frac{r^{\frac{1}{2}}}{C} L_f L^{\frac{1}{2}} + \frac{\sigma^*}{C} + L_g L^{\frac{1}{2}} \right) < 1,$$

we derive that

$$\begin{aligned}
&\|(u_h^{n+1} - \widehat{u}_h^{n+1}, w_h^{n+1} - \widehat{w}_h^{n+1})\|_{\mathcal{H}_h^1 \times L_h^2} \\
&\quad \leq \delta t \left(\frac{r^{\frac{1}{2}}}{C} L_f L^{\frac{1}{2}} + \frac{\sigma^*}{C} + L_g L^{\frac{1}{2}} \right) \|(u_h^{n+1} - \widehat{u}_h^{n+1}, w_h^{n+1} - \widehat{w}_h^{n+1})\|_{\mathcal{H}_h^1 \times L_h^2}.
\end{aligned}$$

It also reads

$$0 \leq \left(1 - \delta t \left(\frac{r^{\frac{1}{2}}}{C} L_f L^{\frac{1}{2}} + \frac{\sigma^*}{C} + L_g L^{\frac{1}{2}} \right) \right) \|(u_h^{n+1} - \widehat{u}_h^{n+1}, w_h^{n+1} - \widehat{w}_h^{n+1})\|_{\mathcal{H}_h^1 \times L_h^2} < 0,$$

with $1 - \delta t \left(\frac{r^{\frac{1}{2}}}{C} L_f L^{\frac{1}{2}} + \frac{\sigma^*}{C} + L_g L^{\frac{1}{2}} \right) > 0$.

Thus, the discrete variational system (3.9) admits a unique solution under the condition (3.18). □

4 Construction of a $T-$periodic sequence $(u_{\delta t,h}, w_{\delta t,h})_{h>0,\delta t>0}$.

In this part, we prove the existence of a $T-$ periodic sequence $(u_{\delta t,h}, w_{\delta t,h})_{h>0,\delta t>0}$ to the discrete variational formulation (4.7). For this purpose, we first establish in Lemma 4 the existence of a specific initial condition (u_h^0, w_h^0) using Brouwer's fixed point theorem. Then, given this initial condition, we construct the $T-$periodic sequence $(u_{\delta t,h}, w_{\delta t,h})_{h>0,\delta t>0}$ thanks to a periodic expansion argument.

4.1 Construction of the initial condition (u_h^0, w_h^0)

We demonstrate the existence of an initial condition (u_h^0, w_h^0) satisfying $(u_h^0, w_h^0) = (u_h^{N_T}, w_h^{N_T})$ using Brouwer's fixed point theorem.

Lemma 4. *Let s be a $T-$ periodic function in $L^\infty(0,T)$. We fix $r > 0$ and $h > 0$. We write Φ for the mapping*

$$(u_h^0, w_h^0) \in L_h^2 \times L_h^2 \mapsto \Phi(u_h^0, w_h^0) := (u_h^{N_T}, w_h^{N_T}) \in L_h^2 \times L_h^2,$$

and introduce the convex, closed, bounded subset

$$B_R^h := \{(u_h, w_h) \in L_h^2 \times L_h^2 \ s.t \ \left(r\|u_h\|_{L^2(\mathcal{I})}^2 + \|w_h\|_{L^2(\mathcal{I})}^2\right)^{\frac{1}{2}} \leq R\},$$

where the radius R satisfies $R^2 := \dfrac{R_1}{R_2}$ with

$$\begin{aligned} R_1 &= \min(C_1 p, 2C_2), \\ R_2 &= 2L\max(-C_0, 0) + 4(p-2)r^2 LC_1 + r\left(\frac{1}{LC^2 C_1 p} + \frac{L^2}{C\sigma_*}\right)\|s\|_{L^\infty(0,T)}^2. \end{aligned}$$

Then, the function Φ admits a fixed point $(u_h^0, w_h^0) \in B_R^h$, which satisfies

$$(u_h^0, w_h^0) = \Phi(u_h^0, w_h^0) = (u_h^{N_T}, w_h^{N_T}).$$

Proof. Let $r > 0$ and $h > 0$. The continuous function Φ is invariant on the subset B_R^h; that is

$$(u_h^0, w_h^0) \in B_R^h \to \Phi(u_h^0, w_h^0) = (u_h^{N_T}, w_h^{N_T}) \in B_R^h.$$

Indeed, we consider the discrete variational formulation (3.9) for $(v_h, z_h) = (u_h^{n+1}, w_h^{n+1})$. Multiplying by r the equation (3.9a) and summing the resulting equality with (3.9b) implies the following

$$\begin{aligned} r\int_{\mathcal{I}} &(u_h^{n+1}(x) - u_h^n(x))u_h^{n+1}(x)\ dx + \int_{\mathcal{I}}(w_h^{n+1}(x) - w_h^n(x))w_h^{n+1}(x)\ dx \\ &+ \frac{r\delta t}{C}\int_{\mathcal{I}}\sigma_h(x)\partial_x^h u_h^{n+1}(x)\partial_x^h u_h^{n+1}(x)\ dx = \frac{r\delta t}{C}s^{n+1}u_{M+1}^{n+1} \\ &- \delta t\int_{\mathcal{I}}\left(\frac{r}{C}f(u_h^{n+1}(x), w_h^{n+1}(x))u_h^{n+1}(x) - g(u_h^{n+1}(x), w_h^{n+1}(x))w_h^{n+1}(x)\right)\ dx. \end{aligned} \tag{4.1}$$

Given the inequality (2.4) in A_3, we get

$$\begin{aligned} \int_{\mathcal{I}}\frac{r}{C}&f(u(t,x), w(t,x))u(t,x) - g(u(t,x), w(t,x))w(t,x)\ dx \\ &\geq C_0 L + C_1\|u(t)\|_{L^p(\mathcal{I})}^p + C_2\|w(t)\|_{L^2(\mathcal{I})}^2. \end{aligned} \tag{4.2}$$

Furthermore, we also deduce by using Young's inequality

$$\frac{C_1 p\beta}{2}\|u_h^{n+1}\|_{L^2(\mathcal{I})}^2 \leq C_1\|u_h^{n+1}\|_{L^p(\mathcal{I})}^p + \frac{(p-2)\beta^2 LC_1}{2}. \tag{4.3}$$

Applying the inequalities (3.17) and (4.2) with the assumption A_1 to the equation (4.1) implies

$$\begin{aligned} \frac{1}{2}&\left(r\|u_h^{n+1}\|_{L^2(\mathcal{I})}^2 + \|w_h^{n+1}\|_{L^2(\mathcal{I})}^2\right) - \frac{1}{2}\left(r\|u_h^n\|_{L^2(\mathcal{I})}^2 + \|w_h^n\|_{L^2(\mathcal{I})}^2\right) \\ &+ \frac{r\delta t\sigma_*}{C}\|\partial_x^h u_h^{n+1}\|_{L^2(\mathcal{I})}^2 + C_1\delta t\|u_h^{n+1}\|_{L^p(\mathcal{I})}^p + C_0\delta t L + C_2\delta t\|w_h^{n+1}\|_{L^2(\mathcal{I})}^2 \\ &\leq \frac{r\delta t}{2}\left(\frac{1}{LC^2\epsilon} + \frac{L^2}{C\theta}\right)\|s\|_{L^\infty(0,T)}^2 + \frac{r\delta t\epsilon}{2}\|u_h^{n+1}\|_{L^2(\mathcal{I})}^2 + \frac{r\delta t\theta}{2C}\|\partial_x^h u_h^{n+1}\|_{L^2(\mathcal{I})}^2. \end{aligned}$$

Let set $\theta = \sigma_*, \epsilon = C_1 p$ and $\beta = 2r$ and we apply (4.3) to get

$$\left(r\|u_h^{n+1}\|^2_{L^2(\mathcal{I})} + \|w_h^{n+1}\|^2_{L^2(\mathcal{I})}\right) - \left(r\|u_h^{n}\|^2_{L^2(\mathcal{I})} + \|w_h^{n}\|^2_{L^2(\mathcal{I})}\right) + \frac{r\delta t\sigma_*}{C}\|\partial_x^h u_h^{n+1}\|^2_{L^2(\mathcal{I})}$$
$$+ rC_1 p\delta t\|u_h^{n+1}\|^2_{L^2(\mathcal{I})} + 2C_0\delta t L + 2C_2\delta t\|w_h^{n+1}\|^2_{L^2(\mathcal{I})} \le r\delta t\left(\frac{1}{LC^2C_1p} + \frac{L^2}{C\sigma_*}\right)\|s\|^2_{L^\infty(0,T)}$$
$$+ \delta t 4(p-2)r^2 LC_1.$$

We write $y_h^{n+1} = r\|u_h^{n+1}\|^2_{L^2(\mathcal{I})} + \|w_h^{n+1}\|^2_{L^2(\mathcal{I})}$ and we denote by R_1 and R_2, the positive constants such that

$$R_1 = \min\left(C_1p, 2C_2\right), \quad R_2 = 2L\max(-C_0, 0) + 4(p-2)r^2LC_1 + r\left(\frac{1}{LC^2C_1p} + \frac{L^2}{C\sigma_*}\right)\|s\|^2_{L^\infty(0,T)}. \quad (4.4)$$

Consequently, it follows that

$$(1 + \delta t R_1)\, y_h^{n+1} - y_h^n \le \delta t R_2.$$

The inequality can also be written as follows

$$y_h^{n+1} \le \frac{1}{1+\delta t R_1} y_h^n + \delta t \frac{R_2}{1+\delta t R_1},$$

with $1 + \delta t R_1 > 0$.

Setting $\alpha := \dfrac{1}{1+\delta t R_1}$ with $\alpha < 1$, we derive

$$y_h^{n+1} \le \alpha y_h^n + \delta t \alpha R_2.$$

Thus, the induction over n implies that $\forall n \in \mathbb{N}$

$$y_h^n \le \alpha^n y_h^0 + \delta t R_2 \frac{\alpha}{\alpha - 1}(\alpha^n - 1),$$

We note that $\dfrac{\alpha}{\alpha-1} = -\dfrac{1}{R_1\delta t}$. Given that $y_h^0 \le R^2$, we deduce

$$y_h^n \le \alpha^n y_h^0 - \frac{R_2}{R_1}(\alpha^n - 1) \le \alpha^n R^2 + R^2(1 - \alpha^n) = R^2,$$

for all $n \ge 0, s.t.\ t_n = n\ \delta t \le N_T \delta t = T$.

The previous result is in particular true for $n = N_T$. Then, we obtain

$$\|\Phi(u_h^0, w_h^0)\|_{L_h^2 \times L_h^2} = \|(u_h^{N_T}, w_h^{N_T})\|_{L_h^2\times L_h^2} = \left(r\|u_h^{N_T}\|^2_{L^2(\mathcal{I})} + \|w_h^{N_T}\|^2_{L^2(\mathcal{I})}\right)^{\frac{1}{2}} \le R.$$

consequently, the function Φ is invariant and admits by means of Brouwer's fixed-point theorem, a fixed point (u_h^0, w_h^0) satisfying $(u_h^0, w_h^0) = (u_h^{N_T}, w_h^{N_T})$. □

4.2 $T-$ periodicity of $(u_{\delta t,h}, w_{\delta t,h})_{h>0,\delta t>0}$

We construct a sequence of $T-$ periodic solutions $(u_{\delta t,h}, w_{\delta t,h})_{h>0,\delta t>0}$ to the space-time discrete variational formulation (4.7). The approach consists in associating the initial condition (u_h^0, w_h^0) obtained in Lemma 4 to the system (4.7). Then, the $T-$ periodicity of $(u_{\delta t,h}, w_{\delta t,h})_{h>0,\delta t>0}$ is established through a periodic expansion argument.

First, we introduce the space-time discrete variational formulation (4.7).

Let $u_{\delta t,h} : [0,T] \mapsto L^2(\mathcal{I})$ be a function defined by

$$u_{\delta t,h}(t) := \sum_{n=0}^{N_T-1} u_h^{n+1}\mathcal{X}_{(t_n,t_{n+1}]}(t), \quad \forall 0 < t \le T, \quad u_{\delta t,h}(0) = u_h^0, \quad (4.5)$$

with u_h^{n+1} the function constant per cell K_i, $\forall i \in [\![1, M]\!]$. We recall that $\mathcal{X}_{(t_n,t_{n+1}]}$ is the indicator function over $(t_n, t_{n+1}]$.

We also define the continuous and affine operator in time $\tilde{u}_{\delta t,h} : [0,T] \longmapsto L^2(\mathcal{I})$ by

$$\tilde{u}_{\delta t}(t) := \sum_{n=0}^{N-1} \left(\frac{u_h^{n+1} - u_h^n}{\delta t}(t - t_n) + u_h^n \right) \chi_{]t_n,t_{n+1}]}(t), \text{ if } 0 \le t \le T.$$

This operator satisfies

$$\partial_t \tilde{u}_{\delta t,h}(t) = \sum_{n=0}^{N-1} \frac{u_h^{n+1} - u_h^n}{\delta t} \chi_{]t_n,t_{n+1}[}(t) \quad \text{in } \mathcal{D}'(0,T). \tag{4.6}$$

In the same way, we define $w_{\delta t,h}$, $v_{\delta t,h}$ and $z_{\delta t,h}$.

We write $s_{\delta t}$ and σ_h for the functions respectively constant per sub-interval $(t_n, t_{n+1}]$ and per diamond $D_{i+1/2}$. They satisfy

$$s_{\delta t}(t) := \sum_{n=0}^{N_T-1} s^{n+1} \mathcal{X}_{(t_n,t_{n+1}]}(t), \quad \sigma_h := \sum_{i=0}^{M} \sigma_{i+\frac{1}{2}} \mathcal{X}_{D_{i+\frac{1}{2}}}.$$

Given these definitions , the discrete variational formulation (3.9) is also written as follows

$$\begin{aligned} &\iint_{\mathcal{I}_T} \partial_t \tilde{u}_{\delta t,h}(t,x) v_{\delta t,h}(t,x)\ dx\ dt + \frac{1}{C} \iint_{\mathcal{I}_T} \sigma_h(x) \partial_x^h u_{\delta t,h}(t,x) \partial_x^h v_{\delta t,h}(t,x)\ dx\ dt \\ &\quad + \frac{1}{C} \iint_{\mathcal{I}_T} f(u_{\delta t,h}(t,x)), w_{\delta t,h}(t,x)) v_{\delta t,h}(t,x)\ dx\ dt = \frac{1}{C} \int_0^T s_{\delta t}(t) v_{\delta t}(t,L)\ dt, \\ &\iint_{\mathcal{I}_T} \partial_t \tilde{w}_{\delta t,h}(t,x) z_{\delta t,h}(t,x)\, dx\, dt = \iint_{\mathcal{I}_T} g(u_{\delta t,h}(t,x), w_{\delta t,h}(t,x)) z_{\delta t,h}(t,x)\, dx\, dt. \end{aligned} \tag{4.7}$$

Theorem 2. *Let $s \in L^\infty(0,T)$ be a $T-$ periodic function and let $(u_h^0, w_h^0) = (u_h^{N_T}, w_h^{N_T})$ be the initial condition provided in Lemma 4. Thus, there exists a discrete initial condition $(u_{\delta t,h}(0), w_{\delta t,h}(0))$ satisfying*

$$u_{\delta t,h}(0) = u_{\delta t,h}(T), \quad w_{\delta t,h}(0) = w_{\delta t,h}(T). \tag{4.8}$$

Moreover, the space-time discrete variational formulation (4.7) associated with the initial condition (4.8) admits a sequence of $T-$ periodic solutions $(u_{\delta t,h}, w_{\delta t,h})_{h>0,\delta t>0}$.

Proof. Let $h > 0$ and $\delta t > 0$. Let $(u_h^0, w_h^0) = (u_h^{N_T}, w_h^{N_T})$ be the initial condition provided in Lemma 4. Given the definition of $u_{\delta t,h}$ and $w_{\delta t,h}$, it is trivial that

$$u_{\delta t,h}(0) = u_h^0 = u_h^{N_T} = u_{\delta t,h}(T), \quad w_{\delta t,h}(0) = w_h^0 = w_h^{N_T} = w_{\delta t,h}(T).$$

Further, based on the definition $u_{\delta t,h}$ and $w_{\delta t,h}$ and given Theorem 1, we easily deduce that the discrete problem (4.7) admits a solution $(u_{\delta t,h}, w_{\delta t,h})$ which tends to be unique under the condition (3.18) over δt.

Now, we establish the $T-$ periodicity of the solutions $(u_{\delta t,h}, w_{\delta t,h})$ by employing the periodic expansion argument. The method consists of proving by induction the following

$$u_{\delta t,h}(t) = u_{\delta t,h}(t + kT), \quad w_{\delta t,h}(t) = w_{\delta t,h}(t + kT), \quad \forall t \in [0,T], \forall k \ge 0. \tag{4.9}$$

In fact, we denote by $p_{\delta t,h}^k$ and $q_{\delta t,h}^k$ the functions satisfying

$$p_{\delta t,h}^k(t) = u_{\delta t,h}(t + kT), \quad q_{\delta t,h}^k(t) = w_{\delta t,h}(t + kT), \quad \forall t \in [0,T], \forall k \ge 0.$$

Thus, for $k = 1$, the functions $p_{\delta t,h}^1$ and $q_{\delta t,h}^1$ are solutions to the discrete problem (4.7). Indeed, given the $T-$ periodicity of the function s, we deduce that $p_{\delta t,h}^1$ and $q_{\delta t,h}^1$ verify the equalities of the discrete problem (4.7), that is

$$\iint_{\mathcal{I}_T} \partial_t \tilde{p}^1_{\delta t,h}(t,x) v_{\delta t,h}(t,x)\ dx\ dt + \frac{1}{C}\iint_{\mathcal{I}_T} \sigma_h(x)\partial^h_x p^1_{\delta t,h}(t,x)\partial^h_x v_{\delta t,h}(t,x)\ dx\ dt$$
$$+\frac{1}{C}\iint_{\mathcal{I}_T} f(p^1_{\delta t,h}(t,x), q^1_{\delta t,h}(t,x)) v_{\delta t,h}(t,x)\ dx\ dt = \frac{1}{C}\int_0^T s_{\delta t}(t) v_{\delta t}(t,L)\ dt,$$
$$\iint_{\mathcal{I}_T} \partial_t \tilde{q}^1_{\delta t,h}(t,x) z_{\delta t,h}(t,x)\, dx\, dt, = \iint_{\mathcal{I}_T} g(p^1_{\delta t,h}(t,x), q^1_{\delta t,h}(t,x)) z_{\delta t,h}(t,x)\, dx\, dt,$$

for $v_{\delta t,h}, z_{\delta t,h}$ defined as in (4.5).

Moreover, we have

$$p^1_{\delta t,h}(0) = u_{\delta t,h}(T) = u_{\delta t,h}(0), \quad q^1_{\delta t,h}(0) = w_{\delta t,h}(T) = w_{\delta t,h}(0).$$

Then, we deduce that $(p^1_{\delta t,h}, q^1_{\delta t,h})$ and $(u_{\delta t,h}, w_{\delta t,h})$ are solutions to the discrete variational formulation (4.7) associated with the same initial condition $(u_{\delta t,h}(0), w_{\delta t,h}(0))$. In consequence, these solutions are equal and satisfy

$$u_{\delta t,h}(t) = p^1_{\delta t,h}(t) = u_{\delta t,h}(t+T), \quad w_{\delta t,h}(t) = q^1_{\delta t,h}(t) = w_{\delta t,h}(t+T), \forall t \in [0,T].$$

Now, we assume that for $\forall k \geq 0$, $p^k_{\delta t,h}$ and $q^k_{\delta t,h}$ are solutions to the discrete problem (4.7) and satisfy

$$u_{\delta t,h}(t) = p^k_{\delta t,h}(t) = u_{\delta t,h}(t+kT), \quad w_{\delta t,h}(t) = q^k_{\delta t,h}(t) = w_{\delta t,h}(t+kT), \forall t \in [0,T], \tag{4.10}$$

with

$$u_{\delta t,h}(0) = p^k_{\delta t,h}(0) = u_{\delta t,h}(kT), \quad w_{\delta t,h}(0) = q^k_{\delta t,h}(0) = w_{\delta t,h}(kT). \tag{4.11}$$

We prove that for $k+1$, we have

$$u_{\delta t,h}(t) = p^{k+1}_{\delta t,h}(t), \quad w_{\delta t,h}(t) = q^{k+1}_{\delta t,h}(t), \forall t \in [0,T].$$

Indeed, given that $T \in [0,T]$ and applying the argument (4.10) and (4.11) for $k \geq 0$, we have

$$p^{k+1}_{\delta t,h}(0) = u_{\delta t,h}(T+kT) = u_{\delta t,h}(T) = u_{\delta t,h}(0),$$
$$q^{k+1}_{\delta t,h}(0) = w_{\delta t,h}(T+kT) = w_{\delta t,h}(T) = w_{\delta t,h}(0).$$

Additionally, the functions $p^{k+1}_{\delta t,h}$, $q^{k+1}_{\delta t,h}$ verify the discrete problem (4.7). Then, given that $(p^{k+1}_{\delta t,h}, q^{k+1}_{\delta t,h})$ and $(u_{\delta t,h}, w_{\delta t,h})$ are solutions to the discrete variational formulation (4.7) associated with the same initial condition $(u_{\delta t,h}(0), w_{\delta t,h}(0))$, we deduce $\forall t \in [0,T]$

$$u_{\delta t,h}(t) = p^{k+1}_{\delta t,h}(t) = u_{\delta t,h}(t+(k+1)T), \quad w_{\delta t,h}(t) = q^{k+1}_{\delta t,h}(t) = w_{\delta t,h}(t+(k+1)T).$$

Hence, the equalities in (4.9) hold.

Thus, from the preceding results, we obtain a sequence of $T-$ periodic solutions $(u_{\delta t,h}, w_{\delta t,h})_{\delta t,h>0}$ of the discrete problem (4.7). $\square$

5 Convergence

In this section, we establish a priori estimates and prove the convergence of the $T-$ periodic sequence $(u_{\delta t,h}, w_{\delta t,h})_{\delta t,h>0}$, as $\delta t, h \to 0$, toward a global weak $T-$ periodic solution (u, w) of the continuous problem (2.1).

5.1 A priori estimation

Now, we present the following a priori estimates.

Lemma 5. *Let $r > 0$, $h > 0$ and $n \in [\![0, N_T - 1]\!]$. Let $(u_h^{n+1},\, w_h^{n+1})$ be a solution to the discrete variational formulation (3.9). Then, we have the following estimations*

$$r \sum_{n=0}^{N_T-1} \delta t \|u_h^{n+1}\|^2_{L^2(\mathcal{I})} + \sum_{n=0}^{N_T-1} \delta t \|w_h^{n+1}\|^2_{L^2(\mathcal{I})} \leq T R^2, \tag{5.1a}$$

$$\sup_{1\leq n\leq N_T} \left(r\|u_h^n\|^2_{L^2(\mathcal{I})} + \|w_h^n\|^2_{L^2(\mathcal{I})} \right) \leq \tilde{C}_1, \tag{5.1b}$$

$$2C_1 \sum_{n=0}^{N_T-1} \delta t \|u_h^{n+1}\|^p_{L^p(\mathcal{I})} + \frac{r\sigma_*}{C} \sum_{n=0}^{N_T-1} \delta t \|\partial_x^h u_h^{n+1}\|^2_{L^2(\mathcal{I})} \leq \tilde{C}_1. \tag{5.1c}$$

with $\tilde{C}_1$ being a constant depending on T, R, y_h^0, K_1 and K_2, as introduced in the proof below.

Proof. We consider the discrete variational formulation (3.9) tested with $(u_h^{n+1}, w_h^{n+1}) \in \mathcal{H}_h^1 \times L_h^2$. We multiply the equation (3.9a) by r and we sum the resulting equality with (3.9b) to obtain the following

$$\begin{aligned}
& r \int_{\mathcal{I}} (u_h^{n+1}(x) - u_h^n(x)) u_h^{n+1}(x)\; dx + \int_{\mathcal{I}} (w_h^{n+1}(x) - w_h^n(x)) w_h^{n+1}(x)\; dx \\
& \quad + \frac{r\delta t}{C} \int_{\mathcal{I}} \sigma_h(x) \partial_x^h u_h^{n+1}(x) \partial_x^h u_h^{n+1}(x)\; dx = \frac{r\delta t}{C} s^{n+1} u_{M+1}^{n+1} \\
& \quad - \delta t \int_{\mathcal{I}} \left(\frac{r}{C} f(u_h^{n+1}(x), w_h^{n+1}(x)) u_h^{n+1}(x) - g(u_h^{n+1}(x), w_h^{n+1}(x)) w_h^{n+1}(x) \right)\; dx.
\end{aligned}$$

Using the inequalities (3.17), assumption A_1 and the inequality $a(a-b) \geq \frac{1}{2}\left(a^2 - b^2\right)$, we get

$$\begin{aligned}
& \frac{1}{2}\left(r\|u_h^{n+1}\|^2_{L^2(\mathcal{I})} + \|w_h^{n+1}\|^2_{L^2(\mathcal{I})} \right) - \frac{1}{2}\left(r\|u_h^n\|^2_{L^2(\mathcal{I})} + \|w_h^n\|^2_{L^2(\mathcal{I})} \right) + C_1 \delta t \|u_h^{n+1}\|^p_{L^p(\mathcal{I})} \\
& \quad + \frac{r\delta t \sigma_*}{C} \|\partial_x^h u_h^{n+1}\|^2_{L^2(\mathcal{I})} \leq \delta t (\frac{\epsilon}{2} + 1) \left(r\|u_h^{n+1}\|^2_{L^2(\mathcal{I})} + \|w_h^{n+1}\|^2_{L^2(\mathcal{I})} \right) \\
& \quad + \frac{r\delta t}{2} \left(\frac{1}{LC^2\epsilon} + \frac{L^2}{C\theta} \right) \|s\|^2_{L^\infty(0,T)} + \frac{r\delta t\theta}{2C} \|\partial_x^h u_h^{n+1}\|^2_{L^2(\mathcal{I})} + \delta t |C_0| L.
\end{aligned}$$

Let set $\epsilon = 2$, $\theta = \sigma_*$ and we write $y_h^{n+1} = r\|u_h^{n+1}\|^2_{L^2(\mathcal{I})} + \|w_h^{n+1}\|^2_{L^2(\mathcal{I})}$ to get

$$\begin{aligned}
& y_h^{n+1} - y_h^n + 2C_1 \delta t \|u_h^{n+1}\|^p_{L^p(\mathcal{I})} + \frac{r\delta t \sigma_*}{C} \|\partial_x^h u_h^{n+1}\|^2_{L^2(\mathcal{I})} \\
& \quad \leq 4\delta t y_h^{n+1} + \delta t \left(r \left(\frac{1}{LC^2} + \frac{L^2}{C\sigma_*} \right) \|s\|^2_{L^\infty(0,T)} + 2|C_0| L \right).
\end{aligned} \tag{5.2}$$

Consequently, we derive the following inequality

$$y_h^{n+1} - y_h^n \leq \delta t K_2 y_h^{n+1} + \delta t K_1, \tag{5.3}$$

with K_1 and K_2 satisfying

$$K_1 = r \left(\frac{1}{LC^2} + \frac{L^2}{C\sigma_*} \right) \|s\|^2_{L^\infty(0,T)} + 2|C_0| L \quad \text{and } K_2 = 4. \tag{5.4}$$

By applying the discrete Gronwall theorem represented in [43, 15] for the inequality (5.3), it follows that

$$\text{for } 0 < \delta t \leq \delta t_0 < \frac{1}{K_2}, \quad \forall n \geq 0,\; s.t.\; t_n = n\; \delta t \leq T, \text{ we get } y_h^n \leq R^2, \tag{5.5}$$

with

$$R^2 = \left(r\|u_0\|^2_{L^2(\mathcal{I})} + \|w_0\|^2_{L^2(\mathcal{I})} + \frac{K_1}{K_2} \right) \exp\left(\frac{K_2\; T}{1 - K_2\; \delta t_0} \right).$$

We multiply both sides of the inequality (5.5) by δt and we sum it over $n = 1, \cdots, N_T$. Then, we deduce

$$\sum_{n=0}^{N_T-1} \delta t y_h^{n+1} = \sum_{n=1}^{N_T} \delta t y_h^n \leq TR^2 \Leftrightarrow r \sum_{n=0}^{N_T-1} \delta t \|u_h^{n+1}\|^2_{L^2(\mathcal{I})} + \sum_{n=0}^{N_T-1} \delta t \|w_h^{n+1}\|^2_{L^2(\mathcal{I})} \leq TR^2. \tag{5.6}$$

So, we obtain the first estimation (5.1a).

Let $1 \leq K \leq N_T$. We sum the inequality (5.2) over $n = 0, \cdots, K-1$ to get

$$\begin{aligned}
&\sum_{n=0}^{K-1} (y_h^{n+1} - y_h^n) + \sum_{n=0}^{K-1} 2C_1 \delta t \|u_h^{n+1}\|^p_{L^p(\mathcal{I})} + \sum_{n=0}^{K-1} \frac{r \delta t \sigma_*}{C} \|\partial_x^h u_h^{n+1}\|^2_{L^2(\mathcal{I})} \\
&\quad \leq \sum_{n=0}^{K-1} \delta t K_2 y_h^{n+1} + \sum_{n=0}^{K-1} \delta t K_1,
\end{aligned}$$

with K_1 and K_2 defined in (5.4).

Using the telescoping sum over $y_h^{n+1} - y_h^n$ and given the inequality (5.6), our previous inequality is equivalent to

$$y_h^K + 2C_1 \sum_{n=0}^{K-1} \delta t \|u_h^{n+1}\|^p_{L^p(\mathcal{I})} + \frac{r\sigma_*}{C} \sum_{n=0}^{K-1} \delta t \|\partial_x^h u_h^{n+1}\|^2_{L^2(\mathcal{I})} + \leq y_h^0 + T(K_1 + K_2 R^2).$$

Thus, we have the following inequality

$$\begin{aligned}
&y_h^K + 2C_1 \sum_{n=0}^{K-1} \delta t \|u_h^{n+1}\|^p_{L^p(\mathcal{I})} + \frac{r\sigma_*}{C} \sum_{n=0}^{K-1} \delta t \|\partial_x^h u_h^{n+1}\|^2_{L^2(\mathcal{I})} \\
&\quad \leq \left(r \|u_0\|^2_{L^2(\mathcal{I})} + \|w_0\|^2_{L^2(\mathcal{I})} \right) + T(K_1 + K_2 R^2).
\end{aligned} \tag{5.7}$$

Then, we deduce that

$$\underset{1 \leq K \leq N_T}{sup} \; y_h^K \leq \tilde{C}_1,$$

with $\tilde{C_1} = r\|u_0\|^2_{L^2(\mathcal{I})} + \|w_0\|^2_{L^2(\mathcal{I})} + T(K_1 + K_2 R^2)$. So, the inequality (5.1b) is proved.

Finally, setting $K = N_T$ in (5.7) implies

$$2C_1 \sum_{n=0}^{N_T-1} \delta t \|u_h^{n+1}\|^p_{L^p(\mathcal{I})} + \frac{r\sigma_*}{C} \sum_{n=0}^{N_T-1} \delta t \|\partial_x^h u_h^{n+1}\|^2_{L^2(\mathcal{I})} \leq \tilde{C}_1.$$

Hence, the last estimation (5.1c) is demonstrated. $\square$

5.2 Kolmogorov and compactness estimates.

First, we state the Kolmogorov theorem presented in [20].

Definition 4. *(Kolmogorov compactness Lemma) We consider an open bounded set $\omega \subset \mathbb{R}^N$ with $N \geq 1$,$1 \leq q < \infty$ and $A \subseteq L^q(\omega)$. Then, A is relatively compact in $L^q(\omega)$ if and only if there exists $\{p(u), u \in A\} \subset L^q(\mathbb{R}^N)$ such that*

1. *$p(u) = u$ a.e. on $\omega, \forall u \in A$,*
2. *$\{p(u), u \in A\}$ is bounded in $L^q(\mathbb{R}^N)$,*
3. *$\|p(u)(\cdot + \eta) - p(u)\|_{L^q(\mathbb{R}^N)} \to 0$ as $\eta \to 0$ with respect to $u \in A$.*

Lemma 6. *Let* $\tilde{u}_{\delta t,h} : \mathbb{R} \times \mathbb{R} \mapsto \mathbb{R}$ *be the extension of* $u_{\delta t,h}$ *such that*

$$\tilde{u}_{\delta t,h}(t,x) = \begin{cases} u_{\delta t,h}(t,x), & \text{if } (t,x) \in \mathcal{I}_T, \\ 0, & \text{otherwise.} \end{cases}$$

Thus, its extension $\tilde{u}_{\delta t,h}$ *verifies* $\tilde{u}_{\delta t,h} = u_{\delta t,h}$, *a.e. in* $\mathcal{I}_T$ *and is bounded in* $L^2(\mathbb{R}^{N_T})$, $\forall \delta t > 0, h > 0$, *under the estimation* (5.1a).

Further, $\tilde{u}_{\delta t,h}$ *satisfies the property of time and space translation estimates, for all* $\delta t > 0$ *and* $h > 0$*; that is, for* $\eta \in (0, L)$ *and* $\tau \in (0,T)$, *there exist constants* $\mu_1 > 0, \mu_2 > 0$ *such that*

$$\iint_{(0,T)\times\mathbb{R}} |\tilde{u}_{\delta t,h}(t,x+\eta) - \tilde{u}_{\delta t,h}(t,x)|^2 \, dx \, dt \leq \eta(\eta + 2h)\mu_1, \tag{5.8a}$$

$$\iint_{(0,T-\tau)\times\mathbb{R}} |\tilde{u}_{\delta t,h}(t+\tau,x) - \tilde{u}_{\delta t,h}(t,x)|^2 \, dx \, dt \leq \tau\mu_2, \tag{5.8b}$$

The proof of this lemma is inspired by the book [20].

5.3 Convergence

We demonstrate the existence of a global weak $T-$ periodic solution (u, w) of the continuous problem (2.1).

We state the following convergence lemma.

Lemma 7. *We consider the sequence of* $T-$ *periodic solutions* $(u_{\delta t,h}, w_{\delta t,h})_{h>0,\delta t>0}$ *obtained in Theorem 2. Based on Lemma 6 and the Kolmogorov Definition 4, there exists a subsequence* $(u_{\delta t,h})_{\delta t,h>0}$ *such that*

$$u_{\delta t,h} \underset{\delta t,h\to 0}{\longrightarrow} u, \quad \text{strongly in } L^2(\mathcal{I}_T). \tag{5.9}$$

It also satisfies

$$u_{\delta t,h} \underset{\delta t,h\to 0}{\rightharpoonup} u, \quad \text{in } L^2(\mathcal{I}_T). \tag{5.10}$$

Furthermore, given the estimation (5.1a), *there also exists a subsequence* $(w_{\delta t,h})_{\delta t,h>0}$ *verifying*

$$w_{\delta t,h} \underset{\delta t,h\to 0}{\rightharpoonup} w, \quad \text{in } L^2(\mathcal{I}_T). \tag{5.11}$$

Theorem 3. *The subsequence* $(u_{\delta t,h}, w_{\delta t,h})_{h>0,\delta t>0}$ *converges to the weak periodic solution* (u, w) *of system* (2.1) *in the sense of Definition 1.*

Proof. For simplicity, we abbreviate the term Dominated Convergence Theorem as D.C.T.

Let $v_h = \varphi_h(t_n, \cdot) = \varphi_h^n$ and $z_h = \xi_h(t_n, \cdot) = \xi_h^n$. Then, given the equation (3.9), we obtain

$$\begin{aligned} C \sum_{n=0}^{N_T-1} \sum_{i=1}^{M} h_i (u_i^{n+1} - u_i^n)\varphi_i^n + \sum_{n=0}^{N_T-1} \delta t \sum_{i=0}^{M+1} F_{i+\frac{1}{2}}^{n+1} (\varphi_{i+1}^n - \varphi_i^n) \\ + \sum_{n=0}^{N_T-1} \delta t \sum_{i=1}^{M} h_i f(u_i^{n+1}, w_i^{n+1})\varphi_i^n = \sum_{n=0}^{N_T-1} \delta t s^{n+1} \varphi_{M+1}^n, \end{aligned} \tag{5.12a}$$

$$\sum_{n=0}^{N_T-1} \sum_{i=1}^{M} h_i (w_i^{n+1} - w_i^n)\xi_i^n = \sum_{n=0}^{N_T-1} \delta t \sum_{i=1}^{M} h_i g(u_i^{n+1}, w_i^{n+1})\xi_i^n. \tag{5.12b}$$

By the telescoping series, we derive

$$\sum_{n=0}^{N_T-1} \sum_{i=1}^{M} h_i (u_i^{n+1} - u_i^n)\varphi_i^n = - \sum_{n=0}^{N_T-1} \sum_{i=1}^{M} h_i u_i^{n+1} (\varphi_i^{n+1} - \varphi_i^n) + \sum_{i=1}^{M} h_i \left(u_i^{N_T} - u_i^0\right) \varphi_i^0,$$

with $\varphi_i^0 = \varphi_i^{N_T}$, for all $i \in [\![1, M]\!]$.

In the same way, we obtain

$$\sum_{n=0}^{N_T-1} \sum_{i=1}^{M} h_i (w_i^{n+1} - w_i^n)\xi_i^n = - \sum_{n=0}^{N_T-1} \sum_{i=1}^{M} h_i w_i^{n+1}(\xi_i^{n+1} - \xi_i^n) + \sum_{i=1}^{M} h_i \left(w_i^{N_T} - w_i^0 \right) \xi_i^0,$$

with $\xi_i^0 = \xi_i^{N_T}$, for all $i \in [\![1, M]\!]$.

Given that $(u_h^0, w_h^0) = (u_h^{N_T}, w_h^{N_T})$, we derive that

$$\begin{aligned}
\sum_{n=0}^{N_T-1} \sum_{i=1}^{M} h_i (u_i^{n+1} - u_i^n)\varphi_i^n &= - \sum_{n=0}^{N_T-1} \sum_{i=1}^{M} h_i u_i^{n+1}(\varphi_i^{n+1} - \varphi_i^n),\\
\sum_{n=0}^{N_T-1} \sum_{i=1}^{M} h_i (w_i^{n+1} - w_i^n)\xi_i^n &= - \sum_{n=0}^{N_T-1} \sum_{i=1}^{M} h_i w_i^{n+1}(\xi_i^{n+1} - \xi_i^n).
\end{aligned}$$

Consequently, the equalities (5.12a) and (5.12b) are also expressed as follows

$$\begin{aligned}
&- C \sum_{n=0}^{N_T-1} \delta t \sum_{i=1}^{M} h_i u_i^{n+1} \left(\frac{\varphi_i^{n+1} - \varphi_i^n}{\delta t} \right) + \sum_{n=0}^{N_T-1} \delta t \sum_{i=0}^{M} h_{i+1/2}\sigma_{i+\frac{1}{2}} \left(\frac{u_{i+1}^{n+1} - u_i^{n+1}}{h_{i+1/2}} \right) \left(\frac{\varphi_{i+1}^n - \varphi_i^n}{h_{i+1/2}} \right)\\
&\quad + \sum_{n=0}^{N_T-1} \delta t \sum_{i=1}^{M} h_i f(u_i^{n+1}, w_i^{n+1})\varphi_i^n = \sum_{n=0}^{N_T-1} \delta t s^{n+1} \varphi_{M+1}^n,
\end{aligned} \tag{5.13a}$$

$$- \sum_{n=0}^{N_T-1} \delta t \sum_{i=1}^{M} h_i w_i^{n+1} \left(\frac{\xi_i^{n+1} - \xi_i^n}{\delta t} \right) = \sum_{n=0}^{N_T-1} \delta t \sum_{i=1}^{M} h_i g(u_i^{n+1}, w_i^{n+1})\xi_i^n. \tag{5.13b}$$

Let $\varphi_{\delta t,h}$, $\xi_{\delta t,h}$, $\tilde{\psi}_{\delta t,h}$, $\bar{\psi}_{\delta t,h}$ and $\tilde{\rho}_{\delta t,h}$ be the functions constant per $(t_n, t_{n+1}] \times K_i$ satisfying

$$\begin{aligned}
&\varphi_{\delta t,h}|_{(t_n,t_{n+1}]\times K_i} = \varphi_i^n = \varphi(t^n, x_i), \quad \xi_{\delta t,h}|_{(t_n,t_{n+1}]\times K_i} = \xi_i^n = \xi(t^n, x_i),\\
&\tilde{\psi}_{\delta t,h}|_{(t_n,t_{n+1}]\times K_i} = \frac{\varphi_i^{n+1} - \varphi_i^n}{\delta t}, \quad \bar{\psi}_{\delta t,h}|_{(t_n,t_{n+1}]\times K_i} = \frac{\varphi_{i+1}^n - \varphi_i^n}{h_{i+1/2}},\\
&\tilde{\rho}_{\delta t,h}|_{(t_n,t_{n+1}]\times K_i} = \frac{(\xi_i^{n+1} - \xi_i^n)}{\delta t},
\end{aligned}$$

$\forall i \in [\![1, M]\!]$, $\forall n \in [\![0, N_T - 1]\!]$.

Then, the system of equations (5.13) is written as follows

$$\begin{aligned}
&- C \iint_{\mathcal{I}_T} u_{\delta t,h}(t, x)\tilde{\psi}_{\delta t,h}(t, x)\, dx\, dt + \iint_{\mathcal{I}_T} f(u_{\delta t,h}(t, x), w_{\delta t,h}(t, x))\varphi_{\delta t,h}(t, x)\, dx\, dt\\
&\quad + \iint_{\mathcal{I}_T} \sigma_h(x)\partial_x^h u_{\delta t,h}(t, x)\bar{\psi}_{\delta t,h}(t, x)\, dx\, dt = \int_0^T s_{\delta t}(t)\varphi_{\delta t}(t, L)\, dt,\\
&- \iint_{\mathcal{I}_T} w_{\delta t,h}(t, x)\tilde{\rho}_{\delta t,h}(t, x)\, dx\, dt = \iint_{\mathcal{I}_T} g(u_{\delta t,h}(t, x), w_{\delta t,h}(t, x))\xi_{\delta t,h}(t, x)\, dx\, dt.
\end{aligned}$$

The strong convergence

$$\tilde{\psi}_{\delta t,h} \underset{\delta t,h\to 0}{\longrightarrow} \partial_t \varphi, \quad in\ L^2(\mathcal{I}_T), \quad \tilde{\rho}_{\delta t,h} \underset{\delta t,h\to 0}{\longrightarrow} \partial_t \xi, \quad in\ L^2(\mathcal{I}_T),$$

together with the convergences (5.9) and (5.11), is sufficient to pass to the limit in the linear terms. We focus now on the non-linear terms.

First, we have from Assumption A_2, the expressions:

$$\begin{aligned}
f(u_{\delta t,h}, w_{\delta t,h}) &= f_1(u_{\delta t,h}) + (f_2(u_{\delta t,h}) - f_2(u))w_{\delta t,h} + f_2(u)w_{\delta t,h},\\
g(u_{\delta t,h}, w_{\delta t,h}) &= g_1(u_{\delta t,h}) + g_2 w_{\delta t,h}.
\end{aligned} \tag{5.14}$$

The continuity of f_1 and g_1 (see Assumption A_2) and the convergence (5.9) lead to

$$f_1(u_{\delta t,h}) \underset{\delta t,h\to 0}{\longrightarrow} f_1(u) \text{ and } g_1(u_{\delta t,h}) \underset{\delta t,h\to 0}{\longrightarrow} g_1(u) \quad a.e. \ \in \mathcal{I}_T.$$

Moreover, the function f_1 and g_1 are respectively uniformly bounded in $L^{p'}(\mathcal{I}_T)$ and $L^2(\mathcal{I}_T)$. Then, we deduce

$$f_1(u_{\delta t,h}) \underset{\delta t,h\to 0}{\rightharpoonup} f_1(u) \quad \text{weakly in } L^{p'}(\mathcal{I}_T) \text{ and } g_1(u_{\delta t,h}) \underset{\delta t,h\to 0}{\rightharpoonup} g_1(u) \text{ weakly in } L^2(\mathcal{I}_T). \tag{5.15}$$

In addition, using the D.C.T., we can easily prove that

$$\varphi_{\delta t,h} \underset{\delta t,h\to 0}{\longrightarrow} \varphi \ in\ L^p(\mathcal{I}_T) \text{ and } \xi_{\delta t,h} \underset{\delta t,h\to 0}{\longrightarrow} \xi \ in\ L^2(\mathcal{I}_T). \tag{5.16}$$

Now, based on the convergences in (5.15) and (5.16), we get

$$\begin{aligned}
&\iint_{\mathcal{I}_T} f_1(u_{\delta t,h}(t,x))\varphi_{\delta t,h}(t,x)\,dx\,dt \underset{\delta t,h\to 0}{\longrightarrow} \iint_{\mathcal{I}_T} f_1(u(t,x))\varphi(t,x)\,dx\,dt,\\
&\iint_{\mathcal{I}_T} g_1(u_{\delta t,h}(t,x))\xi_{\delta t,h}(t,x)\,dx\,dt \underset{\delta t,h\to 0}{\longrightarrow} \iint_{\mathcal{I}_T} g_1(u(t,x))\xi(t,x)\,dx\,dt,\\
&\iint_{\mathcal{I}_T} g_2 w_{\delta t,h}(t,x)\xi_{\delta t,h}(t,x)\,dx\,dt \underset{\delta t,h\to 0}{\longrightarrow} \iint_{\mathcal{I}_T} g_2 w(t,x)\xi(t,x)\,dx\,dt.
\end{aligned} \tag{5.17}$$

Further, since $f_2(u)\varphi_{\delta t,h} \in L^2(\mathcal{I}_T)$ and given the following strong convergence

$$f_2(u)\varphi_{\delta t,h} \underset{\delta t,h\to 0}{\longrightarrow} f_2(u)\varphi \ in\ L^2(\mathcal{I}_T),$$

we get

$$\iint_{\mathcal{I}_T} f_2(u(t,x))w_{\delta t,h}(t,x)\varphi_{\delta t,h}(t,x)\,dx\,dt \underset{\delta t,h\to 0}{\longrightarrow} \iint_{\mathcal{I}_T} f_2(u(t,x))w(t,x)\varphi(t,x)\,dx\,dt, \tag{5.18}$$

Finally, since the functions $f_2(u_{\delta t,h}) - f_2(u)$ and $\varphi_{\delta t,h}$ are respectively uniformly bounded in $L^{2p/p-2}(\mathcal{I}_T)$ and $L^p(\mathcal{I}_T)$, with $\dfrac{2}{2p/p-2} + \dfrac{2}{p} = 1$, then the inequality below holds

$$\begin{aligned}
&\iint_{\mathcal{I}_T} (f_2(u_{\delta t,h}(t,x)) - f_2(u(t,x)))w_{\delta t,h}(t,x)\varphi_{\delta t,h}(t,x)\,dx\,dt\\
&\qquad \leq \|(f_2(u_{\delta t,h}) - f_2(u))\varphi_{\delta t,h}\|_{L^2(\mathcal{I}_T)}\|w_{\delta t,h}\|_{L^2(\mathcal{I}_T)},
\end{aligned}$$

with $w_{\delta t,h}$ uniformly bounded in $L^2(\mathcal{I}_T)$.

One one hand, given (5.9) and because the continuous function $(f_2(u_{\delta t,h}) - f_2(u))^2$ is uniformly bounded in $L^{p/p-2}(\mathcal{I}_T)$, we get

$$(f_2(u_{\delta t,h}) - f_2(u))^2 \underset{\delta t,h\to 0}{\rightharpoonup} 0 \text{ weakly in } L^{p/p-2}(\mathcal{I}_T). \tag{5.19}$$

On the other hand, based on D.C.T., the function $\varphi^2_{\delta t,h} \in L^{p/2}(\mathcal{I}_T)$ strongly converges to φ^2 with

$$\varphi^2_{\delta t,h} \underset{\delta t,h\to 0}{\longrightarrow} \varphi^2 \ in\ L^{p/2}(\mathcal{I}_T). \tag{5.20}$$

Then, from (5.19) and (5.20), we obtain

$$\|(f_2(u_{\delta t,h}) - f_2(u))\varphi_{\delta t,h}\|_{L^2(\mathcal{I}_T)} \underset{\delta t,h\to 0}{\longrightarrow} 0, \tag{5.21}$$

and we derive

$$\iint_{\mathcal{I}_T} (f_2(u_{\delta t,h}(t,x)) - f_2(u(t,x)))w_{\delta t,h}(t,x)\varphi_{\delta t,h}(t,x)\,dx\,dt \underset{\delta t,h\to 0}{\longrightarrow} 0. \tag{5.22}$$

We conclude from the previous results that the sequence $(u_{\delta t,h}, w_{\delta t,h})_{\delta t,h}$ converges to a weak solution (u, w) satisfying the continuous system (2.6a).

Now, we show that

$$u(0, x) = u(T, x), \qquad w(0, x) = w(T, x), \forall x \in \mathcal{I}.$$

Let φ, ξ in $\mathcal{D}((0, T) \times \mathcal{I})$, the weak formulation (2.6a) is then equivalent to the following distribution form

$$\begin{aligned}
&\langle -Cu, \partial_t \varphi\rangle_{\mathcal{D}'\times\mathcal{D}} + \langle f(u, w), \varphi\rangle_{\mathcal{D}'\times\mathcal{D}} + \langle \sigma\partial_x u, \partial_x \varphi\rangle_{\mathcal{D}'\times\mathcal{D}} = 0,\\
&\langle -w, \partial_t \xi\rangle_{\mathcal{D}'\times\mathcal{D}} = \langle g(u, w), \xi\rangle_{\mathcal{D}'\times\mathcal{D}},
\end{aligned}$$

In consequence, the solution (u, w) satisfies the following equations in $\mathcal{D}'((0, T) \times \mathcal{I})$

$$\begin{aligned}
&C\partial_t u + f(u, w) - \partial_x (\sigma\partial_x u) = 0,\\
&\partial_t w = g(u, w),
\end{aligned} \tag{5.23}$$

Next, we consider the functions $\varphi \in \mathscr{C}^1([0, T], H^1(\mathcal{I}))$ and $\xi \in \mathscr{C}^1([0, T], L^2(\mathcal{I}))$ s.t. $\varphi(0, \cdot) = \varphi(T, \cdot)$ and $\xi(0, \cdot) = \xi(T, \cdot)$. Thus, the partial integrations over (5.23) are expressed by

$$\begin{aligned}
&C\int_{\mathcal{I}} (u(T, x) - u(0, x))\, \varphi(0, x)\, dx - C\iint_{\mathcal{I}_T} u(t, x)\partial_t\varphi(t, x)\, dx\, dt\\
&\quad + \iint_{\mathcal{I}_T} f(u(t, x), w(t, x))\varphi(t, x) dx dt + \iint_{\mathcal{I}_T} \sigma(x)\partial_x u(t, x)\partial_x\varphi(t, x)\, dx\, dt\\
&\quad = \int_0^T s(t)\varphi(t, L) dt,\\
&\int_{\mathcal{I}} (w(T, x) - w(0, x))\, \xi(0, x)\, dx - \iint_{\mathcal{I}_T} w(t, x)\partial_t\xi(t, x)\, dx\, dt\\
&\quad = \iint_{\mathcal{I}_T} g(u(t, x), w(t, x))\xi(t, x) dx dt,
\end{aligned} \tag{5.24}$$

Then, the comparison between the two weak formulations (2.6a) and (5.24) implies that

$$\int_{\mathcal{I}} (u(T, x) - u(0, x))\, \varphi(0, x)\, dx = 0, \qquad \int_{\mathcal{I}} (w(T, x) - w(0, x))\, \xi(0, x)\, dx = 0.$$

In consequence, we deduce that

$$u(0, x) = u(T, x), \quad w(0, x) = w(T, x), \quad \forall x \in \mathcal{I}.$$

In conclusion, we deduce that (u, w) is a $T-$ periodic global weak solution .

□

6 Numerical approximation for the initial condition by Newton's method

In this section, we use a numerical approach based on Newton's method to approximate our initial condition found via Brouwer's fixed point theorem (see Lemma 4). To achieve this, we first write the discrete problem resulting from the finite volume method in the matrix form (6.1) associated with its discrete initial conditions. Then, we apply Newton's method on the final system (6.5) obtained after a specific reorganization to deduce our approximated initial condition $(\boldsymbol{U}_h^0, \boldsymbol{W}_h^0)$.

We first write the finite volume scheme (3.5a)- (3.5d) associated with the discrete initial conditions (3.8) into its matrix form. For this purpose, we introduce the following vectors

$$\begin{aligned}
&\boldsymbol{U}_h^{n+1} := (u_i^{n+1})_{1\le i\le M}^T, \qquad \boldsymbol{W}_h^{n+1} := (w_i^{n+1})_{1\le i\le M}^T, \ \boldsymbol{S}^{n+1} := \left(0, \cdots, 0, s^{n+1}\right)^T,\\
&F(\boldsymbol{U}_h^{n+1}, \boldsymbol{W}_h^{n+1}) := [f(u_i^{n+1}, w_i^{n+1})]_{1\le i\le M}^T, \quad G(\boldsymbol{U}_h^{n+1}, \boldsymbol{W}_h^{n+1}) := [(u_i^{n+1}, w_i^{n+1})]_{1\le i\le M}^T,
\end{aligned}$$

We introduce the tridiagonal, positive definite matrix $\mathbb{C}_h \in \mathbb{R}^{M\times M}$, defined by

$$
\begin{aligned}
&[\mathbb{C}_h]_{i,i-1} = -\tau_{i-\frac{1}{2}}, \quad [\mathbb{C}_h]_{i,i} = \tau_{i-\frac{1}{2}} + \tau_{i+\frac{1}{2}}, \quad [\mathbb{C}_h]_{i,i+1} = -\tau_{i+\frac{1}{2}}, \quad 2 \le i \le M-1,\\
&[\mathbb{C}_h]_{1,1} = \tau_{3/2}, \quad [\mathbb{C}_h]_{1,2} = -\tau_{3/2},\\
&[\mathbb{C}_h]_{M,M-1} = -\tau_{M-1/2}, \quad [\mathbb{C}_h]_{M,M} = \tau_{M-1/2},
\end{aligned}
$$

where $\tau_{i+\frac{1}{2}} = \dfrac{\sigma_{i+\frac{1}{2}}}{h_{i+\frac{1}{2}}}$.

We also introduce the mass matrix $\mathbb{M}_h \in \mathbb{R}^{M,M}$ is defined by

$$
[\mathbb{M}_h]_{1\le i,j\le M} = \big(Ch_i\,\delta_{ij}\big)_{1\le i,j\le M},
$$

with δ_{ij} the Kronecker symbol equal to 1 if $i = j$.

Based on the definitions given above, our discrete problem is written as follows

$$
\left\{
\begin{array}{l}
\mathbb{M}_h(\boldsymbol{U}_h^{n+1} - \boldsymbol{U}_h^n) + \delta t \mathbb{C}_h\, \boldsymbol{U}_h^{n+1} = -\delta t F(\boldsymbol{U}_h^{n+1}, \boldsymbol{W}_h^{n+1}) + \delta t\ \boldsymbol{S}^{n+1},\\
\boldsymbol{W}_h^{n+1} - \boldsymbol{W}_h^n = \delta t\ G(\boldsymbol{U}_h^{n+1}, \boldsymbol{W}_h^{n+1}),
\end{array}
\right.
\tag{6.1}
$$

with the initial conditions

$$
\boldsymbol{U}_h^0 = (u_i^0)_{1\le i\le M}, \qquad \boldsymbol{W}_h^0 = (w_i^0)_{1\le i\le M}. \tag{6.2}
$$

We set $\mathbb{B}_h = \mathbb{M}_h + \delta t \mathbb{C}_h$. Thus, for $n \in [\![0, N_T - 1]\!]$, problem (6.1) reads

$$
\left\{
\begin{array}{l}
\mathbb{B}_h \boldsymbol{U}_h^{n+1} - \mathbb{M}_h \boldsymbol{U}_h^n + \delta t F(\boldsymbol{U}_h^{n+1}, \boldsymbol{W}_h^{n+1}) = \delta t \boldsymbol{S}^{n+1},\\
\boldsymbol{W}_h^{n+1} - \boldsymbol{W}_h^n = \delta t G(\boldsymbol{U}_h^{n+1}, \boldsymbol{W}_h^{n+1}).
\end{array}
\right.
\tag{6.3}
$$

Taking into account the periodicity of the solution with the initial conditions verifying

$$
(\boldsymbol{U}_h^0, \boldsymbol{W}_h^0) = (\boldsymbol{U}_h^{N_T}, \boldsymbol{W}_h^{N_T}),
$$

system (6.3) when $n = N_T - 1$ reads

$$
\begin{array}{l}
\mathbb{B}_h \boldsymbol{U}_h^0 - \mathbb{M}_h \boldsymbol{U}_h^{N_T-1} + \delta t\ F(\boldsymbol{U}_h^0, \boldsymbol{W}_h^0) - \delta t\ \boldsymbol{S}^{N_T} = 0.\\
\boldsymbol{W}_h^0 - \boldsymbol{W}_h^{N_T-1} - \delta t\ G(\boldsymbol{U}_h^0, \boldsymbol{W}_h^0) = 0.
\end{array}
\tag{6.4}
$$

For the sake of clarity, we refer to the Newton unknown vector as

$$
(\tilde{\boldsymbol{U}}, \tilde{\boldsymbol{W}}) \ := \ ((\boldsymbol{U}_h^n)_{0\le n\le N_T-1}, (\boldsymbol{W}_h^n)_{0\le n\le N_T-1}) \in \mathbb{R}^{2MN_T},
$$

which satisfies $(\boldsymbol{U}_h^0, \boldsymbol{W}_h^0) = (\boldsymbol{U}_h^{N_T}, \boldsymbol{W}_h^{N_T})$. Thus, system (6.4) rewrites as the following nonlinear system of algebraic equations : find $(\tilde{\boldsymbol{U}}, \tilde{\boldsymbol{W}}) \in \mathbb{R}^{2MN_T}$

$$
\mathcal{S}(\tilde{\boldsymbol{U}}, \tilde{\boldsymbol{W}}) = 0 \tag{6.5}
$$

We are interested in solving the nonlinear problem (6.5) thanks to the Newton method. let an initial guess $(\tilde{\boldsymbol{U}}^0, \tilde{\boldsymbol{W}}^0) \in \mathbb{R}^{2MN_T}$ be given. At the linearization step $k \ge 1$, one looks for $(\tilde{\boldsymbol{U}}^k, \tilde{\boldsymbol{W}}^k) \in \mathbb{R}^{2MN_T}$ satisfying the linear system of algebraic equations

$$
\mathbb{A}^{k-1}(\tilde{\boldsymbol{U}}^k, \tilde{\boldsymbol{W}}^k) = \boldsymbol{\mathcal{B}}^{k-1} \tag{6.6}
$$

Here, $\mathbb{A}^{k-1} \in \mathbb{R}^{2MN_T, 2MN_T}$ is the Jacobian matrix at the Newton step $k \ge 1$ and $\boldsymbol{\mathcal{B}}^{k-1} \in \mathbb{R}^{2MN_T}$ is the right-hand side vector at the Newton step $k \ge 1$. Concerning the choice of the stopping criterion, we chose a tolerance ε close to the machine precision, and we stop the Newton procedure when the relative linearization residual satisfies:

$$
\frac{\|\mathcal{S}(\tilde{\boldsymbol{U}}^k, \tilde{\boldsymbol{W}}^k)\|_2}{\|\mathcal{S}((\tilde{\boldsymbol{U}}^0, \tilde{\boldsymbol{W}}^0))\|_2} \le \varepsilon. \tag{6.7}
$$

Once the stopping criterion (6.7) is satisfied we set $(\tilde{\boldsymbol{U}}, \tilde{\boldsymbol{W}}) = (\tilde{\boldsymbol{U}}^k, \tilde{\boldsymbol{W}}^k)$. Finally, extracting in the solution the first M lines and next the block corresponding to line $MN_T + 1$ through the line $M(N_T + 1)$ we obtain the desired initial condition vector, which is periodic.

7 Numerical experiments

In the sequel, we numerically validate our theoretical results using three ionic models previously introduced in the literature [8, 24]. We first recall these models together with their corresponding parameters. We then illustrate the periodicity of the approximate solution by examining its temporal and spatial profiles for a fine space mesh and a small time $\delta t > 0$. Moreover, when $\delta t \to 0$ we demonstrate the convergence of the approximate solution toward the periodic solution (u, w) of system (2.1). The spatial domain is a 1D horizontal domain with length $L = 1$. We consider a time period $T = 300\,\text{ms}$ so that the space-time domain is $\mathcal{I}_T = (0, 300) \times (0, 1)$.

We consider three following ionic models:

- The FitzHugh–Nagumo model represented by

$$f(u, w) = bu(u - a)(u - 1) + w, \quad g(u, w) = \epsilon(ku - w),$$

  with $a = 0.1$, $b = 1$, $\epsilon = 0.01, k = 0.3$ and $r = \epsilon k$.

- The Rogers–McCulloch model described by

$$f(u, w) = bu(u - a)(u - 1) + uw, \quad g(u, w) = \epsilon(ku - w),$$

  where $a = 0.1, b = 0.3, \epsilon = 0.01, k = 0.1$ and $r > 0$.

- The Modified Aliev–Panfilov model where

$$f(u, w) = bu(u - a)(u - 1) + uw, \quad g(u, w) = -\epsilon(ku(u - 1 - d) + w),$$

  with $a = 0.1, b = 0.4, d = 0.3, \epsilon = 0.01, k = 0.3$ and $r = \epsilon k$.

The excitation function s is T-periodic with a sinusoidal rhythm, and is defined by

$$s(t) := 0.02 \sin\left(\frac{2\pi t}{T}\right), \qquad \forall t \in \mathbb{R},$$

The tensor σ is a scalar function defined on the interval $[0, 1]$, with

$$\sigma(x) := 1.2042, \qquad x \in [0, 1].$$

We discretize the space time domain using small space and time steps $h = 0.01$ cm and $\delta t = 0.006$ ms.

First, we compute an approximation of the initial condition by applying Newton's method on the interval $[0, 300]$ as described in Section 6 with a tolerance $\varepsilon = 10^{-8}$. Using this initial condition, we then solve the discrete nonlinear system stemming from the finite volume scheme over the extended interval $[0, 900]$ for a total number of space-time unknowns equal to 150001, in order to capture the periodic behavior of the solution.

In Figure 2, the transmembrane potential $u_{\delta t,h}$ and the recovery variable $w_{\delta t,h}$ are displayed at each time step for the three ionic models introduced above: the FitzHugh–Nagumo model (left), the Rogers–McCulloch model (middle), and the modified Aliev–Panfilov model (right). We clearly observe a periodic behavior of the solutions. All three phases of the action potential, namely depolarization, repolarization, and return to the resting state, are consistently reproduced in each cycle of the solution's periodic behavior.

Figure 3 displays the behavior of the solutions at several time values: $t = 0\,\text{ms}$, $t = 300\,\text{ms}$, $t = 600\,\text{ms}$, $t = 900\,\text{ms}$ for the FitzHugh-Nagumo model. We observe that for these 4 time instants the spatial profiles are roughly the same. Thus, our finite volume discretization produces a periodic solution.

Figure 4 illustrates the periodic behavior of the monodomain solution for the Rogers–McCulloch model. As in the FitzHugh-Nagumo model, it shows the solution at different instants: $t = 0$ ms, $t = 300$ ms, $t = 600$ ms, and $t = 900$ ms, highlighting the temporal periodicity.

Finally, Figure 5 displays the solution at the same time instants in the case of the Aliev–Panfilov model, which also exhibits a periodic profile of the monodomain solution according to the time variable.

In the sequel, we show, through Tables 1–3 and Figures 6–8, the convergence of the approximate solution, when $\delta t \to 0$, to a time-periodic solution of the monodomain model. Due to the computational burden, we

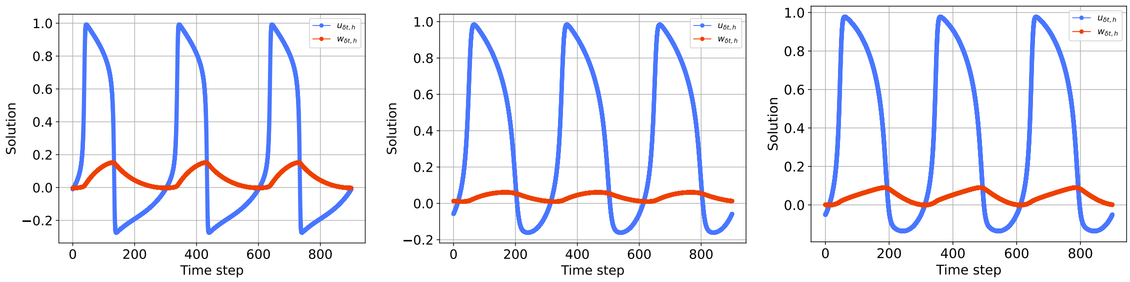

Figure 2: Representation of the temporal profile of the approximate solution $(u_{\delta t,h}, w_{\delta t,h})$ for three ionic models: FitzHugh–Nagumo (left), Rogers–McCulloch (middle), and modified Aliev–Panfilov model (right).

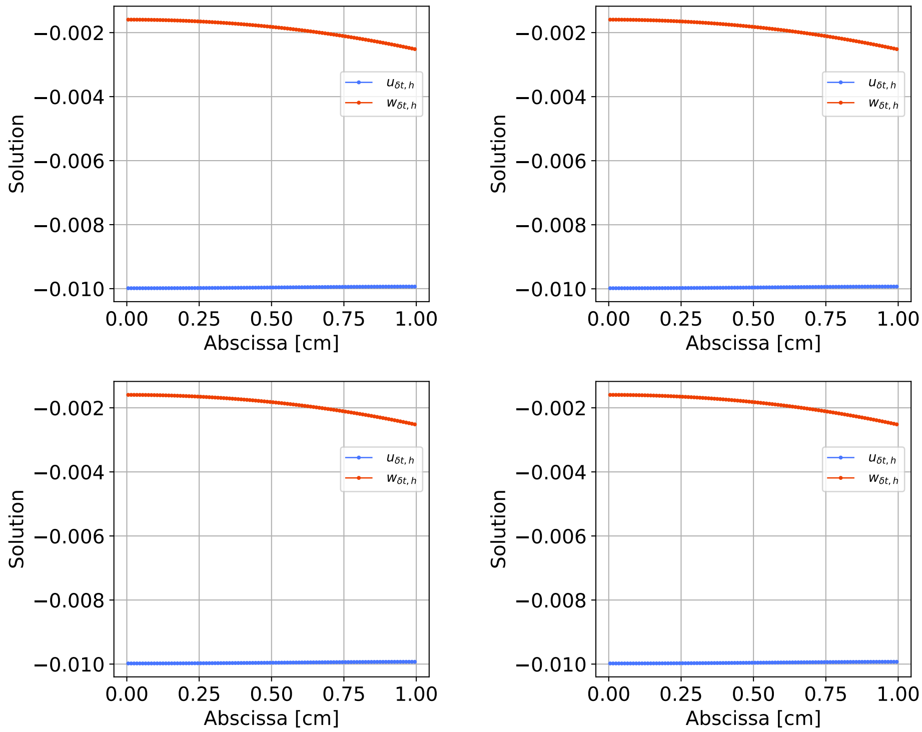

Figure 3: Fitzhugh-Nagumo's solutions at instants $t = 0$ ms (top-left), $t = 300$ ms (top-right), $t = 600$ ms (bottom-left) and $t = 900$ ms (bottom-right).

present the convergence results over the restricted time interval $[0, 1]$. More precisely, in Table 1, we compare the value of the solutions $u_{\delta t,h}$ (top) and $w_{\delta t,h}$ (bottom) at the initial time $t = 0$ with those obtained at the periodic instants 300, 600, and 900 ms, using the $L^2(\mathcal{I})$-norm for different values of δt. For the sake of clarity, for a function v with enough regularity we recall that the $L^2(\mathcal{I})$ norm of v is defined by

$$\|v\|_{L^2(\mathcal{I})} := \left\{ \int_0^T \|v(t)\|_{L^2(0,1)}(t)\,\mathrm{d}t \right\}^{\frac{1}{2}} = \left\{ \int_0^T \int_0^1 |v(t,x)|^2\,\mathrm{d}x\,\mathrm{d}t \right\}^{\frac{1}{2}}. \tag{7.1}$$

It can be observed that, when $\delta t \to 0$, the error tends to zero, indicating that the approximate solution converges to a periodic one. Figure 6 displays the L^2-norm errors of $u_{\delta t,h}$ (left) and $w_{\delta t,h}$ (right) at the periodic instants 300, 600, and 900 ms, compared with the initial instant $t = 0$, clearly illustrating the time-periodic nature of the solution.

Similarly as for the FitzHugh–Nagumo model, Table 2 summarizes the $L^2(\mathcal{I})$-norm errors for both solutions $u_{\delta t,h}$ (top) and $w_{\delta t,h}$ (bottom) in the Rogers-McCulloch model case, with δt tending to zero.

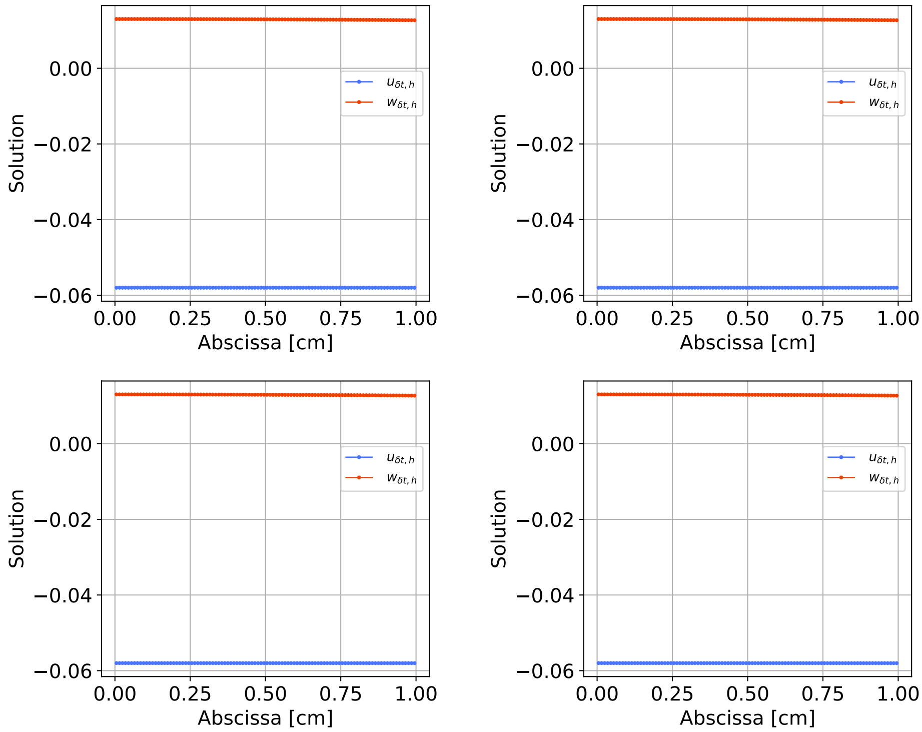

Figure 4: Rogers-McCulloch's solutions at instants $t = 0$ ms (top-left), $t = 300$ ms (top-right), $t = 600$ ms (bottom-left) and $t = 900$ ms (bottom-right).

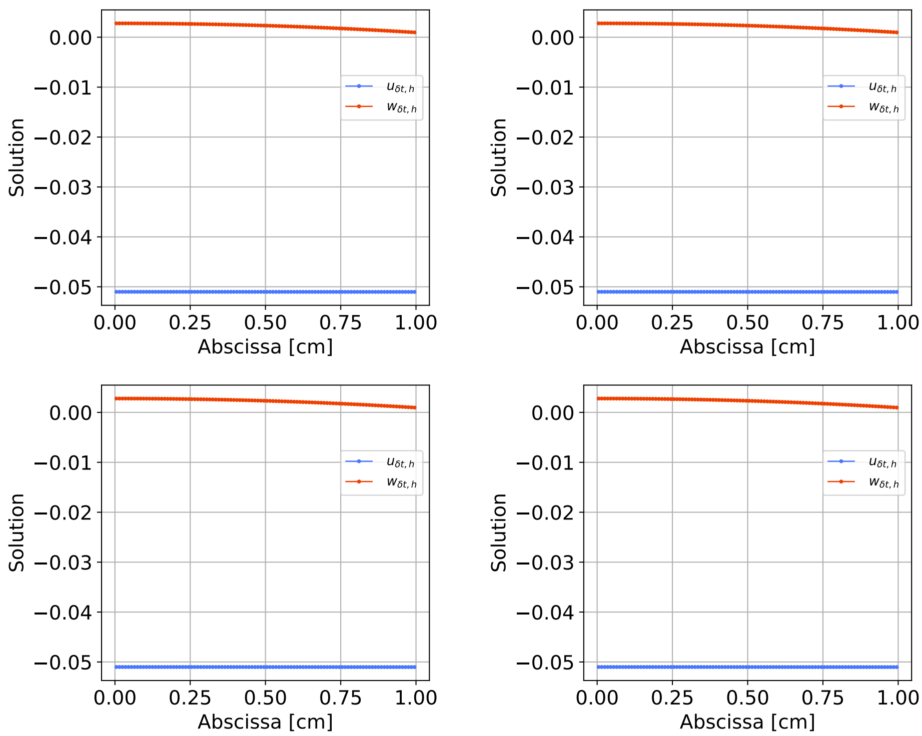

Figure 5: Aliev-panfilov's solutions at instants $t = 0$ ms (top-left), $t = 300$ ms (top-right), $t = 600$ ms (bottom-left) and $t = 900$ ms (bottom-right).

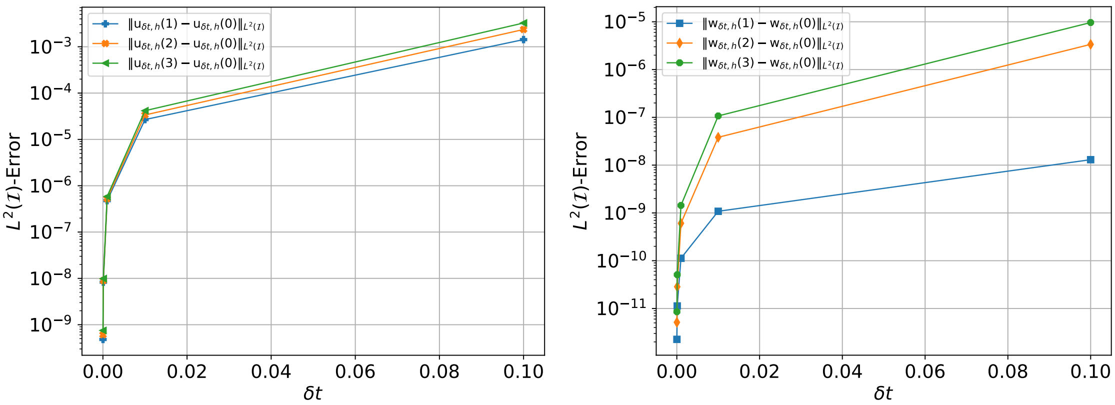

Figure 6: Fitzhugh-Nagumo periodic error's for the solutions $u_{\delta t,h}$ and $w_{\delta t,h}$.

These results are also highlighted in Figure 7, where we observe a fast decreasing behavior of the error to zero. Finally, we also collect in Table 3 the $L^2(\mathcal{I})$-norm errors for both solutions $u_{\delta t,h}$ (top) and $w_{\delta t,h}$ (bottom) for the Aliev-Panfilov model, for a collection of values of δt tending to zero. Figure 8 displays the convergence of the $L^2(\mathcal{I})$-norm errors to zero as δt decreases, highlighting the periodic behavior of the discrete solutions $u_{\delta t,h}$ (left) and $w_{\delta t,h}$ (right) for a small value of δt.

δt	$\|u_{\delta t,h}(0) - u_{\delta t,h}(1)\|_{L^2(\mathcal{I})}$	$\|u_{\delta t,h}(0) - u_{\delta t,h}(2)\|_{L^2(\mathcal{I})}$	$\|u_{\delta t,h}(0) - u_{\delta t,h}(3)\|_{L^2(\mathcal{I})}$
0.1	1.42851094e-03	2.35993472e-03	3.25668565e-03
0.01	2.67971437e-05	3.36845987e-05	4.16121324e-05
0.001	4.76843326e-07	5.24094946e-07	5.85123467e-07
0.0001	8.27621943e-09	8.97857784e-09	9.89142255e-09
2.0e-05	4.80518746e-10	6.07809444e-10	7.51603181e-10

δt	$\|w_{\delta t,h}(0) - w_{\delta t,h}(1)\|_{L^2(\mathcal{I})}$	$\|w_{\delta t,h}(0) - w_{\delta t,h}(2)\|_{L^2(\mathcal{I})}$	$\|w_{\delta t,h}(0) - w_{\delta t,h}(3)\|_{L^2(\mathcal{I})}$
0.1	1.30020279e-08	3.36560675e-06	9.64582135e-06
0.01	1.08184075e-09	3.82232178e-08	1.07095934e-07
0.001	1.12395900e-10	6.06137108e-10	1.43571964e-09
0.0001	1.12853557e-11	2.84931550e-11	5.09084881e-11
2.0e-05	2.25858584e-12	5.13211000e-12	8.54772668e-12

Table 1: $L^2(\mathcal{I})$-norm error between the solution $u_{\delta t,h}$(above) and $w_{\delta t,h}$ (below) at $t = 0$ and its values at instants $t = 1, 2, 3$ ms for different values of δt for the Fitzhugh-Nagumo model.

δt	$\Vert u_{\delta t,h}(0) - u_{\delta t,h}(1)\Vert_{L^2(\mathcal{I})}$	$\Vert u_{\delta t,h}(0) - u_{\delta t,h}(2)\Vert_{L^2(\mathcal{I})}$	$\Vert u_{\delta t,h}(0) - u_{\delta t,h}(3)\Vert_{L^2(\mathcal{I})}$
0.1	1.44644042e-03	2.56158008e-03	3.81676307e-03
0.01	2.68003503e-05	3.48824832e-05	4.57803924e-05
0.001	4.75181022e-07	5.26782807e-07	6.07159262e-07
0.0001	8.58466515e-09	1.02278871e-08	1.26130800e-08
2.0e-05	1.92710156e-09	3.85679097e-09	5.87764584e-09

δt	$\Vert w_{\delta t,h}(0) - w_{\delta t,h}(1)\Vert_{L^2(\mathcal{I})}$	$\Vert w_{\delta t,h}(0) - w_{\delta t,h}(2)\Vert_{L^2(\mathcal{I})}$	$\Vert w_{\delta t,h}(0) - w_{\delta t,h}(3)\Vert_{L^2(\mathcal{I})}$
0.1	3.89997220e-09	1.21737380e-06	3.67418219e-06
0.01	3.32551212e-10	1.34874085e-08	3.96850979e-08
0.001	3.42491544e-11	1.98177174e-10	4.94213005e-10
0.0001	4.22428809e-12	1.15437052e-11	2.20520920e-11
2.0e-05	1.59467172e-12	5.07190195e-12	1.04935322e-11

Table 2: $L^2(\mathcal{I})$-norm error between the solution $u_{\delta t,h}$(above) and $w_{\delta t,h}$ (below) at $t = 0$ and its values at instants $t = 1, 2, 3$ ms for different values of δt for Rogers-McCulloch.

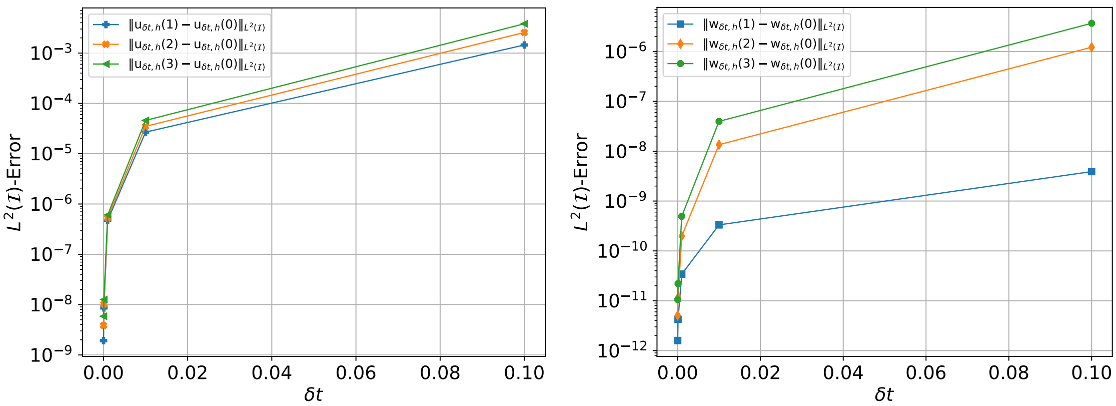

Figure 7: Rogers-McCulloch periodic error's for the solutions $u_{\delta t,h}$ and $w_{\delta t,h}$.

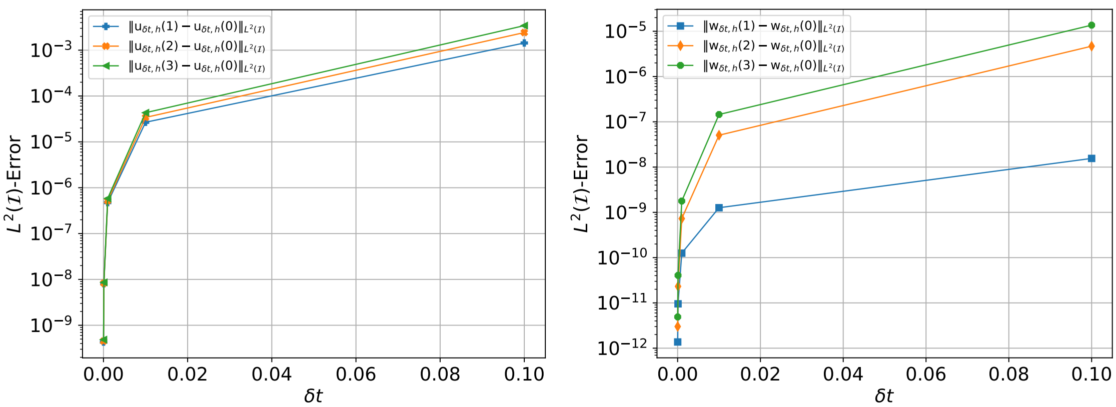

Figure 8: Aliev-panfilov periodic error's for the solutions $u_{\delta t,h}$ and $w_{\delta t,h}$.

δt	$\Vert u_{\delta t,h}(0) - u_{\delta t,h}(1)\Vert_{L^2(\mathcal{I})}$	$\Vert u_{\delta t,h}(0) - u_{\delta t,h}(2)\Vert_{L^2(\mathcal{I})}$	$\Vert u_{\delta t,h}(0) - u_{\delta t,h}(3)\Vert_{L^2(\mathcal{I})}$
0.1	1.43414190e-03	2.42346204e-03	3.42962536e-03
0.01	2.67774367e-05	3.41214592e-05	4.32124483e-05
0.001	4.74477387e-07	5.19497640e-07	5.83296279e-07
0.0001	8.07214464e-09	8.29169969e-09	8.62560342e-09
2.0e-05	4.29925667e-10	4.52506668e-10	4.85890213e-10

δt	$\Vert w_{\delta t,h}(0) - w_{\delta t,h}(1)\Vert_{L^2(\mathcal{I})}$	$\Vert w_{\delta t,h}(0) - w_{\delta t,h}(2)\Vert_{L^2(\mathcal{I})}$	$\Vert w_{\delta t,h}(0) - w_{\delta t,h}(3)\Vert_{L^2(\mathcal{I})}$
0.1	1.55396758e-08	4.68201844e-06	1.36468916e-05
0.01	1.26576119e-09	5.05812451e-08	1.44999072e-07
0.001	1.25652411e-10	7.28057479e-10	1.77862821e-09
0.0001	9.54123388e-12	2.31721993e-11	4.05694254e-11
2.0e-05	1.37005501e-12	3.00850093e-12	4.90071043e-12

Table 3: $L^2(\mathcal{I})$-norm error between the solution $u_{\delta t,h}$(above) and $w_{\delta t,h}$ (below) at $t = 0$ and its values at instants $t = 1, 2, 3$ ms for different values of δt for δt for Aliev-Panfilov.

Acknowledgments

The second author gratefully acknowledges the support of the NASINE project (2023-2025), funded by the PULSAR Academy of Young Researchers in Pays de la Loire, and the research has been partially supported by Centre Henri Lebesgue (CHL), and CNRS International Emerging Action project OTAXIS.

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