Structure preserving reduced order model for a variational inequality

Internship subject jointly proposed by École Supérieure d'Ingénieurs Léonard de Vinci (DVRC, Modelisation Group), École des Ponts ParisTech (CERMICS) and IFP Energies Nouvelles

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1 Scientific context

Variational inequalities are often used to model several physical problems such as contact mechanics (contact problems between several bodies), economy and finance (game theory), and fluid mechanics (multiphase compositional flow with phase transitions). The abstract mathematical formulation of these kind of problems reads: Find $u \in \mathcal{K}_t$ such that $\forall v \in \mathcal{K}_t$

$$\int_0^T \langle \partial_t u_\mu, v - u_\mu \rangle_{V',V}(t)\,\mathrm{d}t + \int_0^T \langle \mathcal{A} u_\mu, v - u_\mu \rangle_{V',V}(t)\,\mathrm{d}t \geq \int_0^T \langle f, v - u_\mu \rangle_{V',V}(t)\,\mathrm{d}t \tag{1}$$

Here, $T > 0$ is the final simulation time, V is a Hilbert space and V' its dual version, $\mathcal{A}$ is a linear continuous coercive operator, f is a source term, $\mathcal{K}_t$ an evolutive -in-time convex set, and $\mu \in \mathcal{P}$ a parameter.

The numerical resolution of such problems is often challenging since it involves a tremendous number of unknowns because of the space-time discretization, and an important number of parameters μ to be tested. More precisely, for each parameter $\mu \in \mathcal{P}$ a space-time problem should be discretized implying thus an unaffordable computational cost and memory burden.

2 Reduced-order-model

Model-order reduction techniques have been developed for many type of problems so as to alleviate the computational burden of parametric studies. In particular they

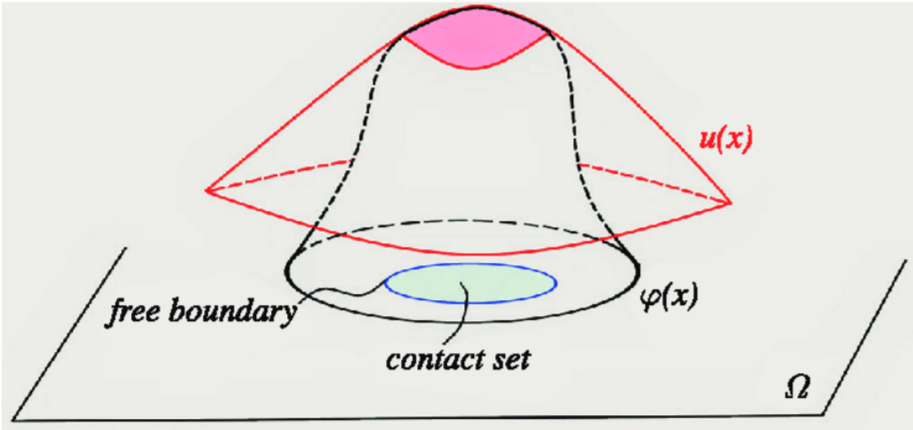

Figure 1: The position u of a membrane in contact with an obstacle fuction φ.

proved to be computationally efficient to approximate the solution of parametrized partial differential equations encountered in many problems in science and engineering. Several methods have been developed in the literature as the Proper Orthogonal Decomposition (POD) method, the Proper Generalized Decomposition (PGD) method, or reduced basis methods. Two main stages constitute the construction of the reduced-order-model procedure. The first one is called the offline stage. This stage is performed once and we compute for some values of $p \in \mathcal{P}_{\text{train}} \subset \mathcal{P}$ high-fidelity solutions called snapshots. We obtain a collection of high-fidelity trajectories

$$\left[\mathbf{U}_{\mu_1}, \mathbf{U}_{\mu_2}, \cdots, \mathbf{U}_{\mu_{\text{card}(\mathcal{P}_{\text{train}})}}\right]$$

Here each $\mathbf{U}_\mu$ is a matrix. From these computed snapshots, we construct a reduced basis in which the reduced solution can be expressed afterwards.

The second stage of this procedure is called the online stage which corresponds to the resolution of the reduced problem for several values of parameters $\mu \in \mathcal{P}$ based on the reduced basis constructed within the offline stage. Therefore, our reduced model enables to approximate the solutions u_μ of (1) with a low computational cost.

3 Objectives of the internship

- First of all, the student will consider a stationary variational inequality in one dimensional space modeling a problem in contact mechanics. Typically, the student will consider a parametrized elliptic obstacle problem in which a membrane is in contact with a fixed surface

$$\begin{aligned} -\boldsymbol{\nabla}\cdot(k\boldsymbol{\nabla}u) - \lambda = f \quad &\text{in} \quad \Omega \\ u-\varphi \geq 0, \quad \lambda \geq 0, \quad (u-\varphi)\lambda = 0 \quad &\text{in} \quad \Omega \\ u = g \quad &\text{on} \quad \partial\Omega. \end{aligned} \tag{2}$$

Here, u is the position of a membrane, f is an external source term and λ is a Lagrange multiplier characterizing the action of the obstacle function φ on the membrane (refer to Figure 1). The function k is here the parameter. The second line of (2) is the linear complementarity conditions saying that either the membrane is above the obstacle or the membrane and the obstacle are in contact. We are interested in finding the optimal value of the parameter k in order to obtain the largest possible contact surface. Solving system (2) for several parameters k is somewhat impossible. To tackle this difficulty, we will construct a reduced-order-model enabling to perform such computations for a wide range of function k. Concerning the numerical resolution of system (2) the student will employ a conforming $\mathbb{P}_1$ finite element discretization. The resulting nonlinear system will be solved by a semismooth Newton procedure.

- The second part of this work will be devoted to construct a POD reduced-order model so as to speed up the computations. This reduced-order model has also be structure preserving in the sense that the numerical reduced solution should preserve the mathematical structure of the continuous model.

- The last part of this internship (depending on the progress of the student) will be dedicated to study a more complex problem such as the storage of radiactive wastes in deep geological layers.

4 General information on the work environment and funding

The internship will be funded by Région Nantes-Pays de-la-Loire (PULSAR academy) and the De Vinci Research Center of École Supérieure d'Ingénieurs Léonard de Vinci (ESILV). The student will be located in ESILV-Nantes and will occasionally travel to IFPEN and CERMICS. The candidate should have a solid background in numerical analysis and PDE.

5 References

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