

A posteriori error estimates in cardiac electrophysiology

ENUMATH Conference 2025

F. Bader, J. Dabaghi, H. Ghazi

Léonard de Vinci Pôle Universitaire, Research Center
92 916 Paris La Défense, France.

September 2, 2025

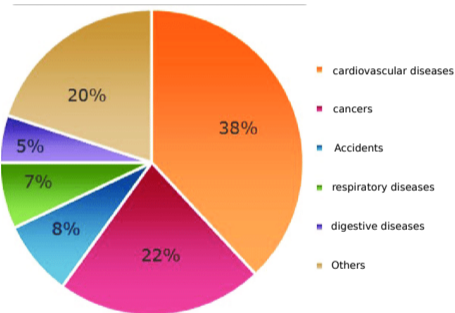
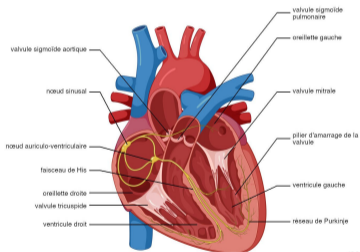


Outline

1. Introduction
2. Model problem
3. Finite element discretization
4. A posteriori error analysis
5. Numerical experiments
6. Conclusion

Introduction

Cardiac electrophysiology studies the electrical signals governing heart contractions and action potential propagation.



We study the monodomain model, a simplified system of PDEs.

Questions : Can we estimate and localize the error within the numerical simulation ?

Monodomain problem

$$\begin{aligned}\partial_t v - \nabla \cdot (\Lambda \nabla v) + f(v, w) &= 0 && \text{in } \Omega \times (0, T), \\ \partial_t w + g(v, w) &= 0 && \text{in } \Omega \times (0, T), \\ \Lambda \nabla v \cdot \mathbf{n} &= 0 && \text{on } \partial\Omega \times (0, T), \\ v|_{t=0} &= v^0, \quad w|_{t=0} = w^0 && \text{in } \Omega.\end{aligned}$$

- unknown variables: **transmembrane potential** $v(t, \mathbf{x}) \in \mathbb{R}$, and the **ionic variable** $w(t, \mathbf{x}) \in \mathbb{R}$ (gating variable, concentration, etc) which is an auxiliary function characterizing the underlying transfer of ions across the membrane.
- ODE : dynamic of the activation and inhibition variables for the ionic channels and $g(v, w)$ is a function representing the ionic activity.

Monodomain problem

$$\begin{aligned}\partial_t v - \nabla \cdot (\Lambda \nabla v) + f(v, w) &= 0 && \text{in } \Omega \times (0, T), \\ \partial_t w + g(v, w) &= 0 && \text{in } \Omega \times (0, T), \\ \Lambda \nabla v \cdot \mathbf{n} &= 0 && \text{on } \partial\Omega \times (0, T), \\ v|_{t=0} &= v^0, \quad w|_{t=0} = w^0 && \text{in } \Omega.\end{aligned}$$

- unknown variables: **transmembrane potential** $v(t, \mathbf{x}) \in \mathbb{R}$, and the **ionic variable** $w(t, \mathbf{x}) \in \mathbb{R}$ (gating variable, concentration, etc) which is an auxiliary function characterizing the underlying transfer of ions across the membrane.
- ODE : dynamic of the activation and inhibition variables for the ionic channels and $g(v, w)$ is a function representing the ionic activity.

Remark

The monodomain problem is a complex coupled nonlinear system of PDEs. Need accurate simulation and localize error within the simulation.

Phenomenological model

Phenomenological model

The phenomenological model: simplifies the underlying mechanisms and focuses on reproducing characteristic behaviors at the macroscopic scale. In terms of CPU cost, these models are competitive but unfortunately do not capture the entire physical processes.

Phenomenological model

The phenomenological model: simplifies the underlying mechanisms and focuses on reproducing characteristic behaviors at the macroscopic scale. In terms of CPU cost, these models are competitive but unfortunately do not capture the entire physical processes.

Examples (FitzHugh-Nagumo)

This model satisfies the Assumptions above

$$\begin{aligned}g(v, w) &= -av + bw, \\f(v, w) &:= f_1(v) + f_2(w)\end{aligned}$$

with

$$f_1(v) := \lambda v(v - 1)(v - \theta) \quad \text{and} \quad f_2(w) = \lambda w$$

and a, b, λ, θ are given parameters satisfying $a \geq 0$, $b \geq 0$, $\lambda > 0$ and $0 < \theta < 1$.

Phenomenological model

The phenomenological model: simplifies the underlying mechanisms and focuses on reproducing characteristic behaviors at the macroscopic scale. In terms of CPU cost, these models are competitive but unfortunately do not capture the entire physical processes.

Examples (FitzHugh-Nagumo)

This model satisfies the Assumptions above

$$\begin{aligned}g(v, w) &= -av + bw, \\f(v, w) &:= f_1(v) + f_2(w)\end{aligned}$$

with

$$f_1(v) := \lambda v(v - 1)(v - \theta) \quad \text{and} \quad f_2(w) = \lambda w$$

and a, b, λ, θ are given parameters satisfying $a \geq 0$, $b \geq 0$, $\lambda > 0$ and $0 < \theta < 1$.

Other possibility : The physiological models which are more realistic but more complicated...

Weak formulation

Given $f \in L^2(0, T; (H^1(\Omega))')$, $g \in L^2(0, T; L^2(\Omega))$, $(v_0, w_0) \in (L^2(\Omega))^2$ and $\Lambda \in L^\infty(\Omega)$, a weak solution of the problem is a vector $\mathbf{U} := (v, w)$ of functions s.t
 $v \in L^2(0, T; H^1(\Omega))$, $\partial_t v \in L^2(0, T; (H^1(\Omega))')$ and $w \in C(0, T; L^2(\Omega))$ verifying

$$\begin{aligned} \frac{d}{dt} \langle v(t), \varphi \rangle_{(H^1(\Omega))', H^1(\Omega)} + (\Lambda \nabla v(t), \nabla \varphi)_\Omega &= - \langle f(v(t), w(t)), \varphi \rangle_{(H^1(\Omega))', H^1(\Omega)} \quad \forall \varphi \in H^1(\Omega) \\ (\partial_t w(t), \phi)_\Omega &= - (g(v(t), w(t)), \phi)_\Omega \quad \forall \phi \in L^2(\Omega). \end{aligned}$$

Theorem (Well-posedness of Monodomain Model)

The monodomain problem admits a unique weak solution.

Finite element method

Time discretization : Implicite Euler scheme

Finite element method

Time discretization : Implicite Euler scheme

Conforming space of piecewise polynomial functions

$$X_h^p := \{v_h \in C^0(\bar{\Omega}); v_h|_K \in \mathbb{P}_p(K) \forall K \in \mathcal{T}_h\} \subset H^1(\Omega)$$

Time discretization : Implicite Euler scheme

Conforming space of piecewise polynomial functions

$$X_h^p := \{v_h \in C^0(\bar{\Omega}); v_h|_K \in \mathbb{P}_p(K) \forall K \in \mathcal{T}_h\} \subset H^1(\Omega)$$

Discrete problem Given $(v_h^0, w_h^0) \in X_h^p \times X_h^p$, a discrete weak formulation consists in searching for all $1 \leq n \leq N_t$, $\mathbf{U}_h^n := (v_h^n; w_h^n) \in X_h^p \times X_h^p$ such that for all φ_h and ϕ_h in X_h^p

$$\begin{aligned} \frac{1}{\Delta t_n} (v_h^n - v_h^{n-1}, \varphi_h)_\Omega + (\Lambda \nabla v_h^n, \nabla \varphi_h)_\Omega &= - (f(v_h^n, w_h^n), \varphi_h)_\Omega \\ \frac{1}{\Delta t_n} (w_h^n - w_h^{n-1}, \phi_h)_\Omega &= - (g(v_h^n, w_h^n), \phi_h)_\Omega, \end{aligned}$$

Time discretization : Implicite Euler scheme

Conforming space of piecewise polynomial functions

$$X_h^p := \{v_h \in C^0(\bar{\Omega}); v_h|_K \in \mathbb{P}_p(K) \forall K \in \mathcal{T}_h\} \subset H^1(\Omega)$$

Discrete problem Given $(v_h^0, w_h^0) \in X_h^p \times X_h^p$, a discrete weak formulation consists in searching for all $1 \leq n \leq N_t$, $\mathbf{U}_h^n := (v_h^n; w_h^n) \in X_h^p \times X_h^p$ such that for all φ_h and ϕ_h in X_h^p

$$\begin{aligned} \frac{1}{\Delta t_n} (v_h^n - v_h^{n-1}, \varphi_h)_\Omega + (\Lambda \nabla v_h^n, \nabla \varphi_h)_\Omega &= - (f(v_h^n, w_h^n), \varphi_h)_\Omega \\ \frac{1}{\Delta t_n} (w_h^n - w_h^{n-1}, \phi_h)_\Omega &= - (g(v_h^n, w_h^n), \phi_h)_\Omega, \end{aligned}$$

Algebraic representation

$$\begin{aligned} \frac{1}{\Delta t_n} \mathbb{M} \mathbf{X}_{1h}^n + \mathbb{S} \mathbf{X}_{1h}^n &= \frac{1}{\Delta t_n} \mathbb{M} \mathbf{X}_{1h}^{n-1} - \mathbf{F}(\mathbf{X}_h^n) \\ \frac{1}{\Delta t_n} \mathbb{M} \mathbf{X}_{2h}^n &= \frac{1}{\Delta t_n} \mathbb{M} \mathbf{X}_{2h}^{n-1} - \mathbf{G}(\mathbf{X}_h^n) \end{aligned}$$

Newton method

For $1 \leq n \leq N_t$ and $\mathbf{X}_h^{n,0} \in \mathbb{R}^{2\mathcal{N}_d^p}$ fixed, typically, $\mathbf{X}_h^{n,0} := \mathbf{X}_h^{n-1}$ where \mathbf{X}_h^{n-1} is the last iterate from the previous time step, one looks at step $k \geq 1$ for $\mathbf{X}_h^{n,k} \in \mathbb{R}^{2\mathcal{N}_d^p}$ such that

$$\mathbb{A}^{n,k-1} \mathbf{X}_h^{n,k} = \mathbf{B}^{n,k-1}.$$

Here,

$$\mathbb{A}^{n,k-1} := \begin{pmatrix} \frac{1}{\Delta t_n} \mathbb{M} + \mathbb{S} + \mathbb{J}_{\tilde{\mathbf{F}}}(\mathbf{X}_{1h}^{n,k-1}) & \lambda \mathbb{M} \\ -a \mathbb{M} & \left(b + \frac{1}{\Delta t_n}\right) \mathbb{M} \end{pmatrix}$$

$$\mathbf{B}^{n,k-1} := \begin{pmatrix} \mathbb{J}_{\tilde{\mathbf{F}}}(\mathbf{X}_{1h}^{n,k-1}) & \lambda \mathbb{M} \\ 0 \mathbb{I}_{\mathcal{N}_d^p, \mathcal{N}_d^p} & 0 \mathbb{I}_{\mathcal{N}_d^p, \mathcal{N}_d^p} \end{pmatrix} \mathbf{X}_h^{n,k-1} + \begin{pmatrix} \frac{1}{\Delta t_n} \mathbb{M} & 0 \mathbb{I}_{\mathcal{N}_d^p, \mathcal{N}_d^p} \\ 0 \mathbb{I}_{\mathcal{N}_d^p, \mathcal{N}_d^p} & \frac{1}{\Delta t_n} \mathbb{M} \end{pmatrix} \mathbf{X}_h^{n-1} - \begin{pmatrix} \mathbf{F}(\mathbf{X}_h^{n,k-1}) \\ \mathbf{0} \end{pmatrix}.$$

Galerkin orthogonality

$$\frac{1}{\Delta t_n} \left(v_h^{n,k} - v_h^{n-1}, \psi_{h,\mathbf{x}_l} \right)_{\Omega} + \left(\Lambda \nabla v_h^{n,k}, \nabla \psi_{h,\mathbf{x}_l} \right)_{\Omega} + \underbrace{\left(\mathcal{F}^{n,k}(v_h^{n,k}, w_h^{n,k}), \psi_{h,\mathbf{x}_l} \right)_{\Omega}}_{\text{linearized ionic terms}} = 0$$

Goal

$$\left\| v - v_{h\tau}^{k} \right\|_{\sharp} + \left\| w - w_{h\tau}^{k} \right\|_b \leq \left(\sum_{n=1}^{N_t} \sum_{K \in \mathcal{T}_h} (\eta_K^n(v_h^{n,k}, w_h^{n,k}))^2 \right)^{\frac{1}{2}}$$

- Fully computable upper bound on the error at each Newton step and each time step
- We employ the methodology of equilibrated flux reconstruction
- Distinguish all error components

A posteriori Analysis

Goal

$$\left\| v - v_{h\tau}^k \right\|_{\sharp} + \left\| w - w_{h\tau}^k \right\|_b \leq \left(\sum_{n=1}^{N_t} \sum_{K \in \mathcal{T}_h} (\eta_K^n(v_h^{n,k}, w_h^{n,k}))^2 \right)^{\frac{1}{2}}$$

- Fully computable upper bound on the error at each Newton step and each time step
- We employ the methodology of equilibrated flux reconstruction
- Distinguish all error components

Space-time representation of the solution (piecewise affine and continuous in time)

$$v_{h\tau}^k|_{I_n} := \frac{v_h^{n,k} - w_h^{n-1}}{\Delta t_n} (t - t^n) + v_h^{n,k} \quad \forall 1 \leq n \leq N_t \quad \text{where} \quad v_{h\tau}^{n,k} := v_{h\tau}^k|_{I_n}$$

Braess & Schoberl (2009), Ern & Vohralík (2013), Dabaghi & Martin Vohralík (2020)

Equilibrated flux reconstruction

Recall :

$$\begin{aligned}\partial_t v - \nabla \cdot (\Lambda \nabla v) + f(v, w) &= 0 && \text{in } \Omega \times (0, T), \\ \partial_t w + g(v, w) &= 0 && \text{in } \Omega \times (0, T), \\ \Lambda \nabla v \cdot \mathbf{n} &= 0 && \text{on } \partial\Omega \times (0, T), \\ v|_{t=0} = v^0, \quad w|_{t=0} = w^0 &&& \text{in } \Omega.\end{aligned}$$

Remark

Violation of physical properties

- $\Lambda \nabla v_{h\tau} \notin L^2(0, T; \mathbf{H}(\text{div}, \Omega))$
- $\nabla \cdot (\Lambda \nabla v_{h\tau}) \neq f(v_{h\tau}, w_{h\tau}) + \partial_t v_{h\tau}$ and $\partial_t w_{h\tau} + g(v_{h\tau}, w_{h\tau}) \neq 0$

We construct a flux $\sigma_{h\tau} \in L^2(0, T; \mathbf{H}(\text{div}, \Omega))$ which "corrects" the violations.

$$\sigma_h^{n,k} \nearrow \left\{ \begin{array}{l} \sigma_{h,\text{disc}}^{n,k} \in \mathbf{H}(\text{div}, \Omega) \\ \sigma_{h,\text{lin}}^{n,k} \in \mathbf{H}(\text{div}, \Omega) \end{array} \right.$$

Procedure of reconstruction of the flux

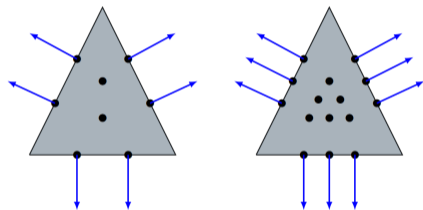
Raviart-Thomas subspaces of $\mathbf{H}(\text{div}, \Omega)$.

$$\mathbf{RT}_p(\Omega) := \{\tau_h \in \mathbf{H}(\text{div}, \Omega), \tau_h|_K \in \mathbf{RT}_p(K) \forall K \in \mathcal{T}_h\} \text{ and } \mathbf{RT}_p(K) := [\mathbb{P}_p(K)]^2 + \vec{x} \mathbb{P}_p(K)$$

number of DOFs $\mathbf{RT}_0(K) = 3$.

number of DOFs $\mathbf{RT}_1(K) = 8$.

number of DOFs $\mathbf{RT}_2(K) = 15$.



Procedure

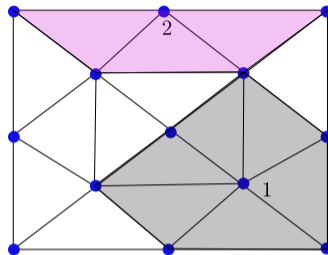
Construct for each vertices of the mesh a flux $\sigma_h^{n,k,a} \in \mathbf{H}(\text{div}, \omega^a)$. The sum

$$\sigma_h^{n,k} = \sum_{a \in \mathcal{V}_h} \sigma_h^{n,k,a} \in \mathbf{H}(\text{div}, \Omega)$$

Mixed finite element system

Find $\sigma_h^{n,k,a} \in V_h^a \subset \mathbf{H}(\text{div}, \omega^a)$ and $\gamma_h^{n,k,a} \in Q_h^a \subset L^2(\omega^a)$ satisfying

$$\begin{aligned} \left(\sigma_h^{n,k,a}, \tau_h \right)_{\omega^a} - \left(\gamma_h^{n,k,a}, \nabla \cdot \tau_h \right)_{\omega^a} &= - \left(\psi_h^a \wedge \nabla v_h^{n,k}, \tau_h \right)_{\omega^a} \\ \left(\nabla \cdot \sigma_h^{n,k,a}, q_h \right)_{\omega^a} &= - \left(\hat{f}_h^{n,k,a}, q_h \right)_{\omega^a} \end{aligned}$$



(MFE) system \iff linear system of algebraic equation. Fast implementation!

$$\sigma_h^{n,k} = \sum_{a \in \mathcal{V}_h} \sigma_h^{n,k,a}$$

Lemma

The total flux reconstruction $\sigma_h^{n,k} \in \mathbf{H}(\text{div}, \Omega)$ and satisfies the equilibration property

$$\left(\nabla \cdot \sigma_h^{n,k} - \mathcal{F}^{n,k}(v_h^{n,k}, w_h^{n,k}) - \partial_t v_h^{n,k}, 1 \right)_K = 0 \quad \forall K \in \mathcal{T}_h$$

A posteriori error estimates

A posteriori error estimates

residual estimators (nonconformity of the flux)

$$\eta_{R_1, K}^{n, k}(t) := C_{P, K} h_K \left\| \partial_t v_{h\tau}^{n, k} + \mathcal{F}(v_{h\tau}^{n, k}, w_{h\tau}^{n, k}) + \nabla \cdot \boldsymbol{\sigma}_{h\tau}^{n, k} \right\|_K(t)$$

$$\eta_{R_2, K}^{n, k}(t) := \left\| \partial_t w_{h\tau}^{n, k} + g(v_{h\tau}^{n, k}, w_{h\tau}^{n, k}) \right\|_K(t)$$

A posteriori error estimates

residual estimators (nonconformity of the flux)

$$\eta_{R_1,K}^{n,k}(t) := C_{P,K} h_K \left\| \partial_t v_{h\tau}^{n,k} + \mathcal{F}(v_{h\tau}^{n,k}, w_{h\tau}^{n,k}) + \nabla \cdot \boldsymbol{\sigma}_{h\tau}^{n,k} \right\|_K(t)$$

$$\eta_{R_2,K}^{n,k}(t) := \left\| \partial_t w_{h\tau}^{n,k} + g(v_{h\tau}^{n,k}, w_{h\tau}^{n,k}) \right\|_K(t)$$

flux estimator (nonconformity of the flux)

$$\eta_{F,K}^{n,k}(t) := \left\| \Lambda \nabla v_{h\tau}^{n,k} + \boldsymbol{\sigma}_{h\tau}^{n,k} \right\|_K(t)$$

A posteriori error estimates

residual estimators (nonconformity of the flux)

$$\eta_{R_1,K}^{n,k}(t) := C_{P,K} h_K \left\| \partial_t v_{h\tau}^{n,k} + \mathcal{F}(v_{h\tau}^{n,k}, w_{h\tau}^{n,k}) + \nabla \cdot \boldsymbol{\sigma}_{h\tau}^{n,k} \right\|_K(t)$$

$$\eta_{R_2,K}^{n,k}(t) := \left\| \partial_t w_{h\tau}^{n,k} + \mathcal{G}(v_{h\tau}^{n,k}, w_{h\tau}^{n,k}) \right\|_K(t)$$

flux estimator (nonconformity of the flux)

$$\eta_{F,K}^{n,k}(t) := \left\| \Lambda \nabla v_{h\tau}^{n,k} + \boldsymbol{\sigma}_{h\tau}^{n,k} \right\|_K(t)$$

linearization estimator (Newton error)

$$\eta_{\text{lin},1,K}^{n,k}(t) := \left\| \mathcal{F}(v_{h\tau}^{n,k}, w_{h\tau}^{n,k}) - f(v_{h\tau}^{n,k}, w_{h\tau}^{n,k}) \right\|_K(t)$$

A posteriori error estimates

residual estimators (nonconformity of the flux)

$$\eta_{R_1,K}^{n,k}(t) := C_{P,K} h_K \left\| \partial_t v_{h\tau}^{n,k} + \mathcal{F}(v_{h\tau}^{n,k}, w_{h\tau}^{n,k}) + \nabla \cdot \sigma_{h\tau}^{n,k} \right\|_K(t)$$

$$\eta_{R_2,K}^{n,k}(t) := \left\| \partial_t w_{h\tau}^{n,k} + g(v_{h\tau}^{n,k}, w_{h\tau}^{n,k}) \right\|_K(t)$$

flux estimator (nonconformity of the flux)

$$\eta_{F,K}^{n,k}(t) := \left\| \Lambda \nabla v_{h\tau}^{n,k} + \sigma_{h\tau}^{n,k} \right\|_K(t)$$

linearization estimator (Newton error)

$$\eta_{lin,1,K}^{n,k}(t) := \left\| \mathcal{F}(v_{h\tau}^{n,k}, w_{h\tau}^{n,k}) - f(v_{h\tau}^{n,k}, w_{h\tau}^{n,k}) \right\|_K(t)$$

initial condition estimators

$$\eta_{IC,1,K}^k := \left\| (v - v_{h\tau}^k)(\cdot, 0) \right\|_K, \quad \eta_{IC,2,K}^k := \left\| (w - w_{h\tau}^k)(\cdot, 0) \right\|_K$$

A posteriori error estimates

Theorem

$$\frac{1}{2\sqrt{C_\star}} \left(\left\| \partial_t \left(w - w_{h\tau}^{n,k} \right) \right\|_Y + \left\| w - w_{h\tau}^{n,k} \right\|_Y + \left\| \partial_t \left(v - v_{h\tau}^{n,k} \right) \right\|_Y + \left\| v - v_{h\tau}^{n,k} \right\|_X \right) \leq \eta^k$$

with the global estimator η^k defined by

$$\eta^k := \left\{ \sum_{n=1}^{N_t} \int_{I_n} \sum_{K \in \mathcal{T}_h} \left(\left(\eta_{R_1,K}^{n,k} + \eta_{F,K}^{n,k} + \eta_{\text{lin},1,K}^{n,k} \right)^2 + \left(\eta_{R_2,K}^{n,k} \right)^2 \right) (t) \, dt \right. \\ \left. + \sum_{K \in \mathcal{T}_h} \left(\left(\eta_{\text{IC},1,K}^k \right)^2 + \left(\eta_{\text{IC},2,K}^k \right)^2 \right) \right\}^{\frac{1}{2}}.$$

Corollary

Let $(v_{h\tau}^k, w_{h\tau}^k)$ be the approximate solution. Let $\sigma_{h\tau}^k \in L^2(0, T; \mathbf{H}(\text{div}, \Omega))$ be the equilibrated flux reconstruction. Then,

$$\eta^k \leq \eta_{\text{tm}}^k + \eta_{\text{FEM}}^k + \eta_{\text{lin}}^k + \eta_{\text{init}}^k.$$

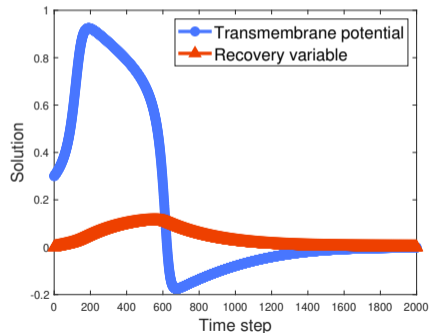
Adaptive Newton algorithm at each time step

1. Choose an initial vector $\mathbf{X}_h^{n,0} \in \mathbb{R}^{2\mathcal{N}_d^p}$, typically as \mathbf{X}_h^{n-1} , and set $k = 1$.
2. From $\mathbf{X}_h^{n,k-1}$ define $\mathbb{A}^{n,k-1} \in \mathbb{R}^{2\mathcal{N}_d^p, 2\mathcal{N}_d^p}$ and $\mathbf{B}^{n,k-1} \in \mathbb{R}^{2\mathcal{N}_d^p}$.
3. Consider the linear system $\mathbb{A}^{n,k-1} \mathbf{X}_h^{n,k} = \mathbf{B}^{n,k-1}$.
4. Compute the estimators and check the stopping criterion $\eta_{\text{lin}}^{n,k} \leq \gamma_{\text{lin}} \eta_{\text{disc}}^{n,k}$. If satisfied, return $\mathbf{X}_h^n = \mathbf{X}_h^{n,k}$. If not, set $k = k + 1$ and go back to 1.

Numerical experiments

First test case at convergence

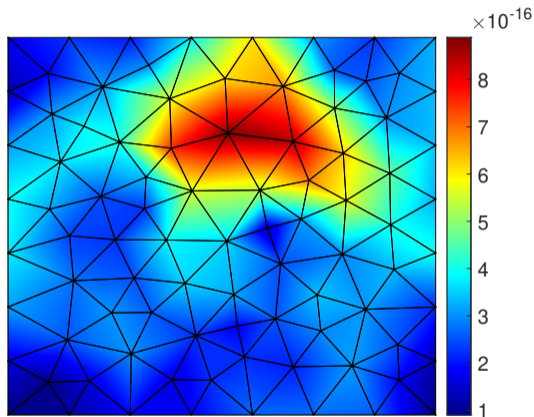
- $\Omega := [-1, 1] \times [-1, 1]$, \mathbb{P}_1 FEM, stopping criterion $\varepsilon = 10^{-12}$, $\Delta t_n = \Delta t = 0.1$
- For modeling ionic current flows, we employ the original FitzHugh–Nagumo model.



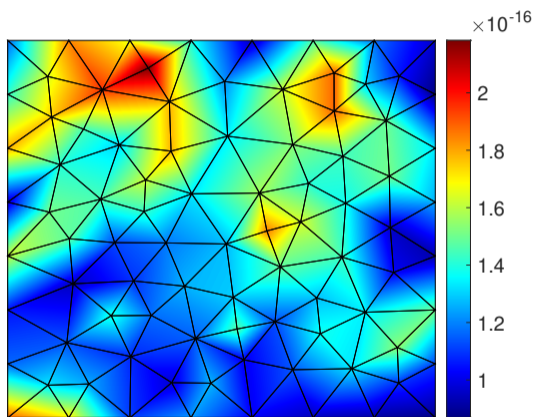
The transmembrane potential v_h (blue curve) represents the electrical activation of a cardiac cell, whereas the recovery variable w_h (red curve) accounts for the delayed response of ionic channels, such as potassium (K^+) and sodium (Na^+) channels. The overall dynamics can be divided into three main phases: rapid depolarization phase, the repolarization phase, return to the resting state phase.

Estimators

Flux estimator

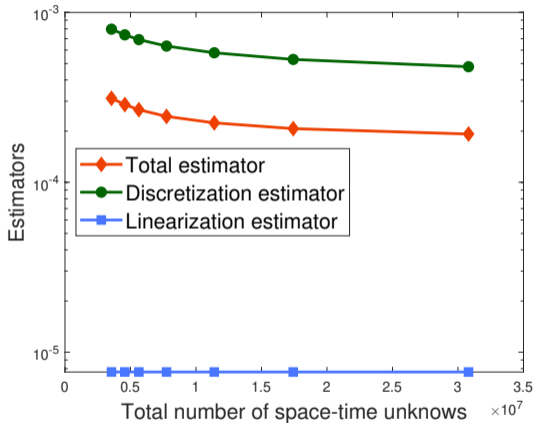
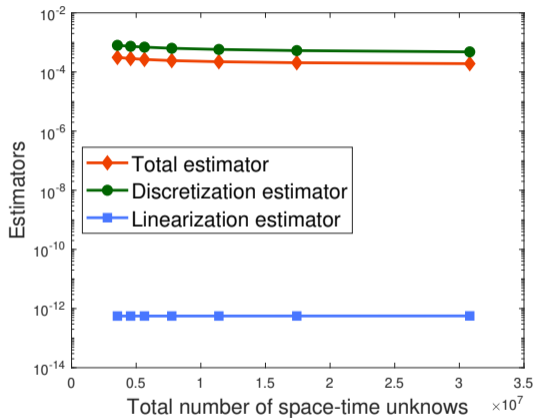


Residual estimator



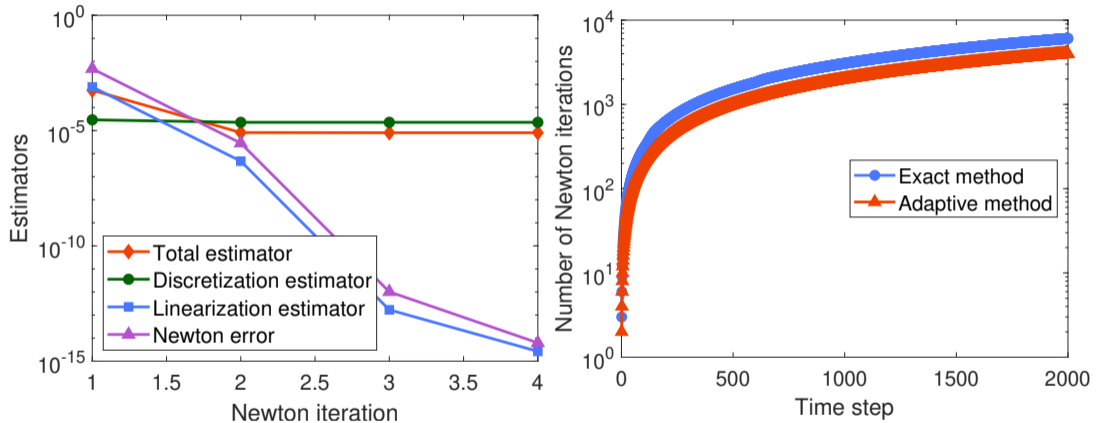
These discretization estimators are very small. When the Newton solver has converged the numerical flux $\nabla v_{h\tau}^{n,k}$ is almost conforming since it is very close in the sense of the L^2 norm to the equilibrated flux $\sigma_{h\tau}^{n,k} \in \mathbf{H}(\text{div}, \Omega)$.

Second test case



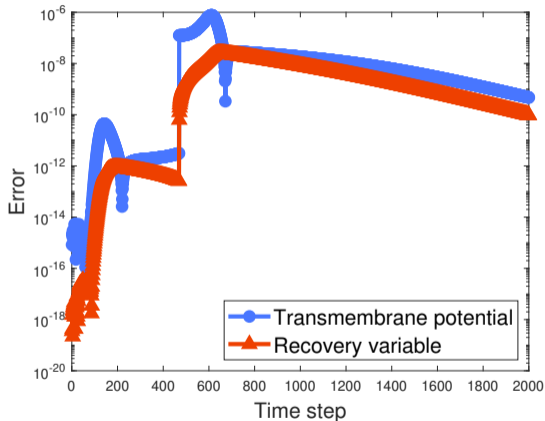
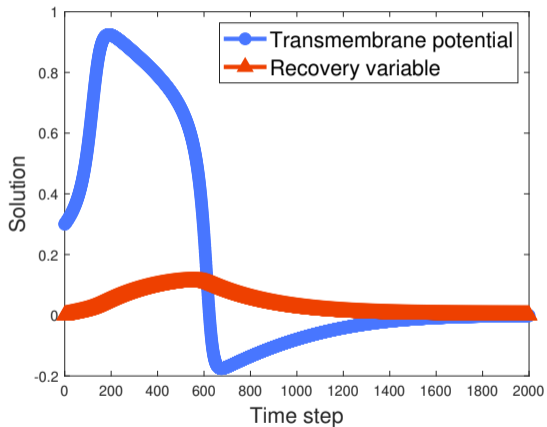
In both cases, the total estimator η^k (red curve) is nearly identical, indicating that the adaptive method achieves comparable accuracy to the exact Newton method.

Adaptivity



Adaptive Newton approach reduces the total number of iterations by approximately 35% by the end of the simulation.

Adaptivity



- In this work we devised a posteriori error estimates with adaptive stopping criteria for the monodomain problem
- Extension to more complicated formulations such as the bidomain problem is under investigation.

F. Bader, J. Dabaghi, and H. Ghazi, *A posteriori error estimates and adaptive stopping criteria for cardiac monodomain model*, submitted for publication, 2025.