Structure-preserving reduced basis method for cross-diffusion systems

JAD DABAGHI, Virginie Ehrlacher

École des Ponts ParisTech (CERMICS) & INRIA Paris (MATHERIALS)

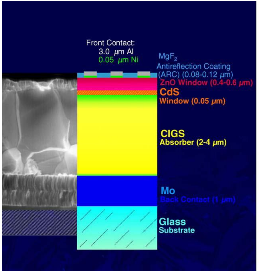
SIAM UQ-22 Conference 12$^{\text{nd}}$ April 2022

Outline

Motivation

Numerical simulation of the PVD process for the fabrication of CIGS (Copper-Indium-Galium-Selenium) solar panels

1. The chemical species are injected under gazeous form in a hot chamber.
2. A cross-diffusion process occurs and the local volumic fraction of the species evolve with respect to time.
3. **goal :** optimize the injected flux to obtain high performance solar cells.



The numerical simulation of the cross-diffusion system is highly expensive.

Need to construct robust schemes to reduce the computational time.

Outline

Model Problem

$\Omega \subset \mathbb{R}^2$: polygonal domain, $T > 0$: final simulation time, N_s : number of chemical species.

Cross-diffusion model

$$\partial_t u_i - \nabla \cdot \left(\sum_{j=1}^{N_s} a_{i,j} \left(u_j \nabla u_i - u_i \nabla u_j \right) \right) = 0 \quad \text{in} \quad \Omega \times [0, T], \text{ for } i \in [1, N_s]\,,$$

$$\left(\sum_{j=1}^{N_s} a_{i,j} \left(u_j \nabla u_i - u_i \nabla u_j \right) \right) \cdot \boldsymbol{n} = 0 \quad \text{on} \quad \partial\Omega \times [0, T], \text{ for } i \in [1, N_s]\,,$$

$$u_i(\boldsymbol{x}, 0) = u_i^0(\boldsymbol{x}) \quad \text{in} \quad \Omega, \text{ for } i \in [1, N_s]\,.$$

- Assume $\mathbb{A} \in \mathbb{R}^{N_s, N_s}$, $\mathbb{A} = (a_{i,j})_{1 \leq i,j \leq N_s}$ is symmetric with nonnegative coefficients and that its diagonal terms vanish.

Gradient flow structure

Entropy functional:

$$E(\boldsymbol{u}) := \int_\Omega \sum_{i=1}^{N_s} u_i(x) \ln(u_i(x))\, \mathrm{d}x \qquad \boldsymbol{u} = (u_i)_{i \in [\![1, N_s]\!]}$$

The cross-diffusion system has a gradient flow structure and can be rewritten as

$$\partial_t \boldsymbol{u} - \boldsymbol{\nabla} \cdot (\mathbb{C}(\boldsymbol{u}) \boldsymbol{\nabla} dE(\boldsymbol{u}))$$
$$(\mathbb{C}(\boldsymbol{u}) \boldsymbol{\nabla} dE(\boldsymbol{u})) \cdot \boldsymbol{n} = 0$$
$$\boldsymbol{u}(\boldsymbol{x}, 0) = \boldsymbol{u}^0(\boldsymbol{x}) \quad \text{in} \quad \Omega.$$

- $\mathbb{C}(\boldsymbol{u}) \in \mathbb{R}^{N_s, N_s}$: mobility matrix
- dE: Entropy differential defined by

$$(dE(\boldsymbol{u}))_i := \frac{\partial E(\boldsymbol{u})}{\partial u_i} = 1 + \ln(u_i).$$

Theorem

There exists a weak solution $\mathbf{u}$ *satisfying*

$$\mathbf{u} \in \left[L^2_{\mathrm{loc}}(\mathbb{R}^+, H^1(\Omega, \mathbb{R}^{N_s}))\right]^{N_s} \quad \textit{and} \quad \partial_t \mathbf{u} \in \left[L^2_{\mathrm{loc}}(\mathbb{R}^+, \left[H^1(\Omega, \mathbb{R}^{N_s})\right]')\right]^{N_s}.$$

Theorem

There exists a weak solution $\boldsymbol{u}$ *satisfying*

$$\boldsymbol{u} \in \left[L^2_{\mathrm{loc}}(\mathbb{R}^+, H^1(\Omega, \mathbb{R}^{N_s}))\right]^{N_s} \quad \textit{and} \quad \partial_t \boldsymbol{u} \in \left[L^2_{\mathrm{loc}}\left(\mathbb{R}^+, \left[H^1(\Omega, \mathbb{R}^{N_s})\right]'\right)\right]^{N_s}.$$

Structural properties of the solution: Consider $\boldsymbol{u}^0 = (u_1^0, \cdots, u_{N_s}^0) \in \mathbb{R}_+^{N_s}$ such that $\sum_{i=1}^{N_s} u_i^0 = 1$ and $\left\|\boldsymbol{u}^0\right\|_{L^\infty(\Omega)} < +\infty$. Then,

Theorem

There exists a weak solution $\boldsymbol{u}$ *satisfying*

$$\boldsymbol{u} \in \left[L^2_{\mathrm{loc}}(\mathbb{R}^+, H^1(\Omega, \mathbb{R}^{N_s}))\right]^{N_s} \quad \text{and} \quad \partial_t \boldsymbol{u} \in \left[L^2_{\mathrm{loc}}(\mathbb{R}^+, \left[H^1(\Omega, \mathbb{R}^{N_s})\right]')\right]^{N_s}.$$

Structural properties of the solution: Consider $\boldsymbol{u}^0 = (u_1^0, \cdots, u_{N_s}^0) \in \mathbb{R}_+^{N_s}$ such that $\sum_{i=1}^{N_s} u_i^0 = 1$ and $\|\boldsymbol{u}^0\|_{L^\infty(\Omega)} < +\infty$. Then,

1. mass conservation: $\int_\Omega u_i(x,t)\,\mathrm{dx} = \int_\Omega u_i{}^0(x)\,\mathrm{dx} \;\; \forall t \in [0,T], \;\; \forall i \in [1, N_s]$.

Theorem

There exists a weak solution $\boldsymbol{u}$ satisfying

$$\boldsymbol{u} \in \left[L^2_{\mathrm{loc}}(\mathbb{R}^+, H^1(\Omega, \mathbb{R}^{N_s}))\right]^{N_s} \quad \textit{and} \quad \partial_t \boldsymbol{u} \in \left[L^2_{\mathrm{loc}}(\mathbb{R}^+, \left[H^1(\Omega, \mathbb{R}^{N_s})\right]')\right]^{N_s}.$$

Structural properties of the solution: Consider $\boldsymbol{u}^0 = (u_1^0, \cdots, u_{N_s}^0) \in \mathbb{R}_+^{N_s}$ such that $\sum_{i=1}^{N_s} u_i^0 = 1$ and $\left\|\boldsymbol{u}^0\right\|_{L^\infty(\Omega)} < +\infty$. Then,

1. mass conservation: $\displaystyle\int_\Omega u_i(x,t)\,\mathrm{dx} = \int_\Omega {u_i}^0(x)\,\mathrm{dx} \;\; \forall t \in [0,T], \;\; \forall i \in [1, N_s]$.
2. positivity: $u_i(x,t) \geq 0 \quad \forall x \in \Omega, \quad \forall t \in [0,T], \quad \forall i \in [1, N_s]$.

Theorem

There exists a weak solution $\boldsymbol{u}$ *satisfying*

$$\boldsymbol{u} \in \left[L^2_{\mathrm{loc}}(\mathbb{R}^+, H^1(\Omega, \mathbb{R}^{N_s}))\right]^{N_s} \quad \textit{and} \quad \partial_t \boldsymbol{u} \in \left[L^2_{\mathrm{loc}}(\mathbb{R}^+, \left[H^1(\Omega, \mathbb{R}^{N_s})\right]')\right]^{N_s}.$$

Structural properties of the solution: Consider $\boldsymbol{u}^0 = (u_1^0, \cdots, u_{N_s}^0) \in \mathbb{R}_+^{N_s}$ such that $\sum_{i=1}^{N_s} u_i^0 = 1$ and $\left\|\boldsymbol{u}^0\right\|_{L^\infty(\Omega)} < +\infty$. Then,

1. mass conservation: $\int_\Omega u_i(x,t)\,\mathrm{dx} = \int_\Omega u_i{}^0(x)\,\mathrm{dx} \;\; \forall t \in [0,T], \;\; \forall i \in [1, N_s]$.
2. positivity: $u_i(x,t) \geq 0 \quad \forall x \in \Omega, \quad \forall t \in [0,T], \quad \forall i \in [1, N_s]$.
3. preservation of the volume filling constraint: $\boldsymbol{u} \in \mathbb{R}_+^{N_s}$ such that $\sum_{i=1}^{N_s} u_i = 1$.

Theorem

There exists a weak solution $\boldsymbol{u}$ *satisfying*

$$\boldsymbol{u} \in \left[L^2_{\text{loc}}(\mathbb{R}^+, H^1(\Omega, \mathbb{R}^{N_s}))\right]^{N_s} \quad \textit{and} \quad \partial_t \boldsymbol{u} \in \left[L^2_{\text{loc}}(\mathbb{R}^+, \left[H^1(\Omega, \mathbb{R}^{N_s})\right]')\right]^{N_s}.$$

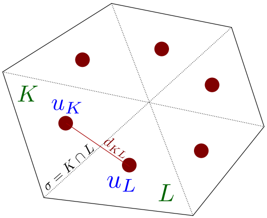
Structural properties of the solution: Consider $\boldsymbol{u}^0 = (u_1^0, \cdots, u_{N_s}^0) \in \mathbb{R}_+^{N_s}$ such that $\sum_{i=1}^{N_s} u_i^0 = 1$ and $\left\|\boldsymbol{u}^0\right\|_{L^\infty(\Omega)} < +\infty$. Then,

1. mass conservation: $\int_\Omega u_i(x,t)\,\mathrm{dx} = \int_\Omega {u_i}^0(x)\,\mathrm{dx} \;\; \forall t \in [0,T], \;\; \forall i \in [1, N_s]$.
2. positivity: $u_i(x,t) \geq 0 \quad \forall x \in \Omega, \quad \forall t \in [0,T], \quad \forall i \in [1, N_s]$.
3. preservation of the volume filling constraint: $\boldsymbol{u} \in \mathbb{R}_+^{N_s}$ such that $\sum_{i=1}^{N_s} u_i = 1$.
4. entropy-entropy dissipation relation

$$\frac{d}{dt}E(\boldsymbol{u}) + \int_\Omega \sum_{1\leq i<j\leq N_s} a_{i,j} u_i(x) u_j(x) |\nabla \ln(u_i(x)) - \nabla \ln(u_j(x))|^2\,\mathrm{dx} = 0.$$

The cell-centered finite Volume method

The cell-centered finite Volume method

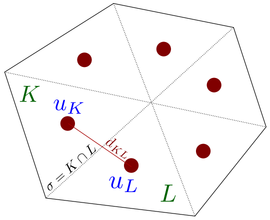


$D_{K\sigma}\mathbf{u}_i^n = u_{i,L}^n - u_{i,K}^n$

- N_s unknowns per cell $\boldsymbol{U}^n := (u_{i,K}^n)_{K\in\mathcal{T}_h, i\in[\![1,N_s]\!]} \in \mathbb{R}^{N_e\times N_s}$
- $\boldsymbol{U}^0 \in \mathbb{R}^{N_s\times N_e}$ where $u_{i,K}^0 = \dfrac{1}{|K|}\displaystyle\int_K {u_i}^0(x)\,\mathrm{dx}$
- FV scheme : find $\boldsymbol{U}^n \in \mathbb{R}^{N_e\times N_s}$ satisfying

$$|K|\frac{u_{i,K}^n - u_{i,K}^{n-1}}{\Delta t_n} + \sum_{\sigma\in\mathcal{E}_K} \mathcal{F}_{i,K\sigma}^n(\boldsymbol{U}^n) = 0$$

The cell-centered finite Volume method



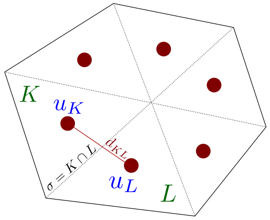
$D_{K\sigma}\mathbf{u}_i^n = u_{i,L}^n - u_{i,K}^n$

- N_s unknowns per cell $\boldsymbol{U}^n := (u_{i,K}^n)_{K\in\mathcal{T}_h, i\in[\![1,N_s]\!]} \in \mathbb{R}^{N_e\times N_s}$
- $\boldsymbol{U}^0 \in \mathbb{R}^{N_s\times N_e}$ where $u_{i,K}^0 = \dfrac{1}{|K|}\displaystyle\int_K {u_i}^0(x)\,\mathrm{d}x$
- FV scheme : find $\boldsymbol{U}^n \in \mathbb{R}^{N_e\times N_s}$ satisfying

$$|K|\frac{u_{i,K}^n - u_{i,K}^{n-1}}{\Delta t_n} + \sum_{\sigma\in\mathcal{E}_K} \mathcal{F}_{i,K\sigma}^n(\boldsymbol{U}^n) = 0$$

Flux: $$\mathcal{F}_{i,K\sigma}^n(\boldsymbol{U}^n) := -a^\star\tau_\sigma D_{K\sigma}\boldsymbol{u}_i^n - \tau_\sigma\left(\sum_{j=1}^{N}(a_{i,j}-a^\star)\left(u_{j,\sigma}^n D_{K\sigma}\boldsymbol{u}_i^n - u_{i,\sigma}^n D_{K\sigma}\boldsymbol{u}_j^n\right)\right).$$

The cell-centered finite Volume method



$D_{K\sigma}\mathbf{u}_i^n = u_{i,L}^n - u_{i,K}^n$

- N_s unknowns per cell $\boldsymbol{U}^n := (u_{i,K}^n)_{K\in\mathcal{T}_h, i\in[\![1,N_s]\!]} \in \mathbb{R}^{N_e\times N_s}$
- $\boldsymbol{U}^0 \in \mathbb{R}^{N_s\times N_e}$ where $u_{i,K}^0 = \dfrac{1}{|K|}\displaystyle\int_K u_i{}^0(x)\,\mathrm{d}x$
- FV scheme : find $\boldsymbol{U}^n \in \mathbb{R}^{N_e\times N_s}$ satisfying

$$|K|\frac{u_{i,K}^n - u_{i,K}^{n-1}}{\Delta t_n} + \sum_{\sigma\in\mathcal{E}_K} \mathcal{F}_{i,K\sigma}^n(\boldsymbol{U}^n) = 0$$

Flux: $$\mathcal{F}_{i,K\sigma}^n(\boldsymbol{U}^n) := -a^\star\tau_\sigma D_{K\sigma}\boldsymbol{u}_i^n - \tau_\sigma\left(\sum_{j=1}^{N}(a_{i,j}-a^\star)\left(u_{j,\sigma}^n D_{K\sigma}\boldsymbol{u}_i^n - u_{i,\sigma}^n D_{K\sigma}\boldsymbol{u}_j^n\right)\right).$$

edge unknown $$u_{i,\sigma}^n := \begin{cases} 0 & \text{if} \quad \min(u_{i,K}^n, u_{i,K\sigma}^n) < 0, \\ u_{i,K}^n & \text{if} \quad u_{i,K}^n = u_{i,K\sigma}^n \geq 0, \\ \dfrac{u_{i,K}^n - u_{i,K\sigma}^n}{\ln(u_{i,K}^n) - \ln(u_{i,K\sigma}^n)} & \text{if} \quad u_{i,K}^n \neq u_{i,K\sigma}^n \geq 0. \end{cases}$$

Remark

- *The main idea of the introduction of the parameter $a^\star > 0$ is to avoid unphysical solutions Cancès, Gaudeul 2020.*
- *The numerical flux is conservative in the sense that for $\sigma \in \mathcal{E}_h^{\text{int}}$, $\sigma = K|L$, $F^n_{i,L\sigma} = -F^n_{i,K\sigma}$.*

Remark

- *The main idea of the introduction of the parameter $a^\star > 0$ is to avoid unphysical solutions Cancès, Gaudeul 2020.*
- *The numerical flux is conservative in the sense that for $\sigma \in \mathcal{E}_h^{\mathrm{int}}$, $\sigma = K|L$, $F^n_{i,L\sigma} = -F^n_{i,K\sigma}$.*

Structural properties of the discrete solution:

Remark

- *The main idea of the introduction of the parameter $a^\star > 0$ is to avoid unphysical solutions Cancès, Gaudeul 2020.*
- *The numerical flux is conservative in the sense that for $\sigma \in \mathcal{E}_h^{\text{int}}$, $\sigma = K|L$, $F^n_{i,L\sigma} = -F^n_{i,K\sigma}$.*

Structural properties of the discrete solution:

Theorem (Cancès, Gaudeul 2020)

1. *mass conservation* $\sum_{K\in\mathcal{T}_h} |K| u^n_{i,K} = \int_\Omega u_i{}^0(x)\,\mathrm{dx} \quad \forall i \in [1, N_s]\,, \quad \forall n \in [0, N_\mathrm{t}]$.
2. *positivity* $u^n_{i,K} > 0 \quad \forall K \in \mathcal{T}_h, \quad \forall i \in [1, N_s]\,, \quad \forall n \in [0, N_\mathrm{t}]$.
3. *Volume filling constraints:* $\sum_{i=1}^{N_s} u^n_{i,K} = 1 \quad \forall K \in \mathcal{T}_h, \quad \forall n \in [0, N_\mathrm{t}]$.
4. *Decays of the discrete entropy* $E_{\mathcal{T}_h}(\boldsymbol{U}^n) \leq E_{\mathcal{T}_h}(\boldsymbol{U}^{n-1}) \quad \forall n \in [1, N_\mathrm{t}]$ *where* $E_{\mathcal{T}_h}(\boldsymbol{U}) := \sum_{K\in\mathcal{T}_h} \sum_{i=1}^{N_s} |K| u_{i,K} \ln(u_{i,K})$.

Newton linearization

The finite volume procedure defines a nonlinear system of algebraic equations

$$G^n(\boldsymbol{U}^n) = 0 \quad \text{where} \quad G^n : \mathbb{R}^{N_e \times N_s} \to \mathbb{R}^{N_e \times N_s}.$$

Newton linearization

The finite volume procedure defines a nonlinear system of algebraic equations

$$G^n(\boldsymbol{U}^n) = 0 \quad \text{where} \quad G^n : \mathbb{R}^{N_e \times N_s} \to \mathbb{R}^{N_e \times N_s}.$$

Initialization of Newton solver: Let $n \in [\![1, N_{\mathrm{t}}]\!]$ and $\boldsymbol{U}^{n,0} \in \mathbb{R}^{N_e \times N_s}$ be fixed (typically $\boldsymbol{U}^{n,0} = \boldsymbol{U}^{n-1}$).

Linear system : the Newton algorithm generates a sequence $(\boldsymbol{U}^{n,k})_{k \geq 1}$, with $\boldsymbol{U}^{n,k} \in \mathbb{R}^{N_e \times N_s}$ solution of

$$\mathbb{A}^{n,k-1}\boldsymbol{U}^{n,k} = \boldsymbol{B}^{n,k-1}.$$

Newton linearization

The finite volume procedure defines a nonlinear system of algebraic equations

$$G^n(\boldsymbol{U}^n) = 0 \quad \text{where} \quad G^n : \mathbb{R}^{N_e \times N_s} \to \mathbb{R}^{N_e \times N_s}.$$

Initialization of Newton solver: Let $n \in [\![1, N_{\mathrm{t}}]\!]$ and $\boldsymbol{U}^{n,0} \in \mathbb{R}^{N_e \times N_s}$ be fixed (typically $\boldsymbol{U}^{n,0} = \boldsymbol{U}^{n-1}$).

Linear system : the Newton algorithm generates a sequence $(\boldsymbol{U}^{n,k})_{k \geq 1}$, with $\boldsymbol{U}^{n,k} \in \mathbb{R}^{N_e \times N_s}$ solution of

$$\mathbb{A}^{n,k-1} \boldsymbol{U}^{n,k} = \boldsymbol{B}^{n,k-1}.$$

The jacobian matrix $\mathbb{A}^{n,k-1} \in \mathbb{R}^{N_e \times N_s, N_e \times N_s}$ and the right-hand side vector $\boldsymbol{B}^{n,k-1} \in \mathbb{R}^{N_e \times N_s}$ are defined by

$$\mathbb{A}^{n,k-1} := \mathbb{J}_{G^n}(\boldsymbol{U}^{n,k-1}) \quad \text{and} \quad \boldsymbol{B}^{n,k-1} := \mathbb{J}_{G^n}(\boldsymbol{U}^{n,k-1})\boldsymbol{U}^{n,k-1} - G^n(\boldsymbol{U}^{n,k-1})$$

Summary

1. We proposed the cell-centered finite volume method to solve the cross-diffusion system.
2. This discrete system preserves the structural properties of the solution.

We want to solve the cross-diffusion problem for a wide variety of cross-diffusion matrices $\mathbb{A}$. It involves high computational cost.

We construct a reduced model to save computational time that preserves the structural properties of the solution.

Outline

The offline stage

The offline stage

Some notation: To each cross-diffusion matrix $\mathbb{A} = (a_{i,j})$ is associated a parameter $\mu \in \mathcal{P}$.

$$\forall \mu \in \mathcal{P} \longleftrightarrow \text{ a solution } \boldsymbol{U}_{\mu}^{n} \in \mathbb{R}^{N_s \times N_e}.$$

The offline stage

Some notation: To each cross-diffusion matrix $\mathbb{A} = (a_{i,j})$ is associated a parameter $\mu \in \mathcal{P}$.

$$\forall \mu \in \mathcal{P} \longleftrightarrow \text{ a solution } \boldsymbol{U}_{\mu}^{n} \in \mathbb{R}^{N_s \times N_e}.$$

The offline stage:

The offline stage

Some notation: To each cross-diffusion matrix $\mathbb{A} = (a_{i,j})$ is associated a parameter $\mu \in \mathcal{P}$.

$$\forall \mu \in \mathcal{P} \longleftrightarrow \text{ a solution } \boldsymbol{U}_\mu^n \in \mathbb{R}^{N_s \times N_e}.$$

The offline stage:

1. We compute snapshots of solution $\boldsymbol{U}_\mu^n \in \mathbb{R}^{N_s \times N_e}$ for $\mu \in \mathcal{P}^{\text{off}} \subset \mathcal{P}$ (a certain number of so-called high-fidelity trajectories). Next, compute the corresponding snapshots matrix

$$\mathbb{M} = \begin{bmatrix} \mathbb{M}_{\mu_1} & \mathbb{M}_{\mu_2} & \cdots & \mathbb{M}_{\mu_{p^\star}} \end{bmatrix} \in \mathbb{R}^{N_s \times N_e, N_t \times p^\star}.$$

The offline stage

Some notation: To each cross-diffusion matrix $\mathbb{A} = (a_{i,j})$ is associated a parameter $\mu \in \mathcal{P}$.

$$\forall \mu \in \mathcal{P} \longleftrightarrow \text{ a solution } \boldsymbol{U}_{\mu}^{n} \in \mathbb{R}^{N_s \times N_e}.$$

The offline stage:

1. We compute snapshots of solution $\boldsymbol{U}_{\mu}^{n} \in \mathbb{R}^{N_s \times N_e}$ for $\mu \in \mathcal{P}^{\text{off}} \subset \mathcal{P}$ (a certain number of so-called high-fidelity trajectories). Next, compute the corresponding snapshots matrix

$$\mathbb{M} = \begin{bmatrix} \mathbb{M}_{\mu_1} & \mathbb{M}_{\mu_2} & \cdots & \mathbb{M}_{\mu_{p^\star}} \end{bmatrix} \in \mathbb{R}^{N_s \times N_e, N_t \times p^\star}.$$

2. SVD decomposition : $\mathbb{M} = \underbrace{\mathbb{V}}_{\in \mathbb{R}^{N_s \times N_e, N_s \times N_e}} \times \underbrace{\mathbb{S}}_{\in \mathbb{R}^{N_s \times N_e, N_t \times p^\star}} \times \underbrace{\mathbb{W}^T}_{\in \mathbb{R}^{N_t \times p^\star, N_t \times p^\star}}.$

   Here, $\mathbb{S}_{ii} = \sqrt{\sigma_i}$ for $1 \leq i \leq \min(N_s \times N_e, N_t \times p^\star)$ and σ_i are the eigenvalues of $\mathbb{M}\mathbb{M}^T$.

The offline stage

Some notation: To each cross-diffusion matrix $\mathbb{A} = (a_{i,j})$ is associated a parameter $\mu \in \mathcal{P}$.

$$\forall \mu \in \mathcal{P} \longleftrightarrow \text{ a solution } \boldsymbol{U}_\mu^n \in \mathbb{R}^{N_s \times N_e}.$$

The offline stage:

1. We compute snapshots of solution $\boldsymbol{U}_\mu^n \in \mathbb{R}^{N_s \times N_e}$ for $\mu \in \mathcal{P}^{\text{off}} \subset \mathcal{P}$ (a certain number of so-called high-fidelity trajectories). Next, compute the corresponding snapshots matrix

$$\mathbb{M} = \begin{bmatrix} \mathbb{M}_{\mu_1} & \mathbb{M}_{\mu_2} & \cdots & \mathbb{M}_{\mu_{p^\star}} \end{bmatrix} \in \mathbb{R}^{N_s \times N_e, N_t \times p^\star}.$$

2. SVD decomposition : $\mathbb{M} = \underbrace{\mathbb{V}}_{\in \mathbb{R}^{N_s \times N_e, N_s \times N_e}} \times \underbrace{\mathbb{S}}_{\in \mathbb{R}^{N_s \times N_e, N_t \times p^\star}} \times \underbrace{\mathbb{W}^T}_{\in \mathbb{R}^{N_t \times p^\star, N_t \times p^\star}}.$

   Here, $\mathbb{S}_{ii} = \sqrt{\sigma_i}$ for $1 \leq i \leq \min(N_s \times N_e, N_t \times p^\star)$ and σ_i are the eigenvalues of $\mathbb{M}\mathbb{M}^T$.

3. Select r columns from the matrice $\mathbb{V}$ as follows : $\sum_{k \geq r+1} \sigma_k^2 \leq \varepsilon$ for $\varepsilon \geq 0$ a fixed tolerance. $\Rightarrow$ We obtain a reduced basis $\mathbb{V}^r = (\boldsymbol{V}_1, \cdots, \boldsymbol{V}_r)$.

The online stage

For each $\mu \in \mathcal{P}$, at each time step $n = 1 \cdots N_t$, the solution of the reduced model denoted by $\widetilde{\boldsymbol{U}}_\mu^n \in \mathbb{R}^{N_s \times N_e}$ is expressed in the basis $(\boldsymbol{V}^1, \cdots, \boldsymbol{V}^r)$ as

$$\widetilde{\boldsymbol{U}}_\mu^n := \sum_{k=1}^{r} c_\mu^{k,n} \boldsymbol{V}^k, \quad \widetilde{\boldsymbol{U}}_\mu^0 := \Pi_{\text{span}(\boldsymbol{V}^1, \cdots, \boldsymbol{V}^r)} \boldsymbol{U}^0.$$

How to derive the expression of the coefficients $c_\mu^{k,n}$?

We define the function $H : \mathbb{R}^r \to \mathbb{R}^r$ by $H_l(\boldsymbol{c}_\mu^n) := \langle \boldsymbol{V}^l, G^n(\widetilde{\boldsymbol{U}}_\mu^n) \rangle \quad \forall 1 \leq l \leq r$.
The vector $\boldsymbol{c}_\mu^n \in \mathbb{R}^r$ is solution to the nonlinear problem

$$H(\boldsymbol{c}_\mu^n) = 0.$$

Remark

This reduced model does not necessarily preserves the structural properties of the numerical solution.

Outline

The offline stage

The offline stage

1. Compute snapshots of solutions.

The offline stage

1. Compute snapshots of solutions.
2. Compute the matrix $\overline{\mathbb{M}} \in \mathbb{R}^{N_s \times N_e, N_t \times p^\star}$ defined by

$$\overline{\mathbb{M}} = \left[\overline{\mathbb{M}}_{\mu_1} \quad \overline{\mathbb{M}}_{\mu_2} \quad \cdots \quad \overline{\mathbb{M}}_{\mu_{p^\star}} \right] \in \mathbb{R}^{N_s \times N_e, N_t \times p^\star}.$$

where each matrix $\overline{\mathbb{M}}_{\mu_\alpha} \in \mathbb{R}^{N_s \times N_e, N_t}$ are defined by $\left[\overline{\mathbb{M}}_{\mu_\alpha}\right]_{i,K} = z^n_{\mu_\alpha,i,K} = \ln(u^n_{\mu_\alpha,i,K})$.

The offline stage

1. Compute snapshots of solutions.
2. Compute the matrix $\overline{\mathbb{M}} \in \mathbb{R}^{N_s \times N_e, N_t \times p^\star}$ defined by

$$\overline{\mathbb{M}} = \left[\overline{\mathbb{M}}_{\mu_1} \quad \overline{\mathbb{M}}_{\mu_2} \quad \cdots \quad \overline{\mathbb{M}}_{\mu_{p^\star}}\right] \in \mathbb{R}^{N_s \times N_e, N_t \times p^\star}.$$

where each matrix $\overline{\mathbb{M}}_{\mu_\alpha} \in \mathbb{R}^{N_s \times N_e, N_t}$ are defined by $\left[\overline{\mathbb{M}}_{\mu_\alpha}\right]_{i,K} = z^n_{\mu_\alpha,i,K} = \ln(u^n_{\mu_\alpha,i,K})$.

3. SVD decomposition

$$\overline{\mathbb{M}} = \underbrace{\overline{\mathbb{V}}}_{\in \mathbb{R}^{N_s \times N_e, N_s \times N_e}} \times \underbrace{\overline{\mathbb{S}}}_{\in \mathbb{R}^{N_s \times N_e, N_t \times p^\star}} \times \underbrace{\overline{\mathbb{W}}^T}_{\in \mathbb{R}^{N_t \times p^\star, N_t \times p^\star}}.$$

The offline stage

1. Compute snapshots of solutions.
2. Compute the matrix $\overline{\mathbb{M}} \in \mathbb{R}^{N_s \times N_e, N_t \times p^\star}$ defined by

$$\overline{\mathbb{M}} = \left[\overline{\mathbb{M}}_{\mu_1} \quad \overline{\mathbb{M}}_{\mu_2} \quad \cdots \quad \overline{\mathbb{M}}_{\mu_{p^\star}} \right] \in \mathbb{R}^{N_s \times N_e, N_t \times p^\star}.$$

   where each matrix $\overline{\mathbb{M}}_{\mu_\alpha} \in \mathbb{R}^{N_s \times N_e, N_t}$ are defined by $\left[\overline{\mathbb{M}}_{\mu_\alpha}\right]_{i,K} = z^n_{\mu_\alpha,i,K} = \ln(u^n_{\mu_\alpha,i,K})$.
3. SVD decomposition

$$\overline{\mathbb{M}} = \underbrace{\overline{\mathbb{V}}}_{\in \mathbb{R}^{N_s \times N_e, N_s \times N_e}} \times \underbrace{\overline{\mathbb{S}}}_{\in \mathbb{R}^{N_s \times N_e, N_t \times p^\star}} \times \underbrace{\overline{\mathbb{W}}^T}_{\in \mathbb{R}^{N_t \times p^\star, N_t \times p^\star}} .$$

4. Select r basis functions. Add to the matrix $\overline{\mathbb{V}}^r$ N_s identity bloc matrices as follows

Example: $N_s = 3$

$$\overline{\mathbb{V}}^{r^\star} = \begin{bmatrix}
\overline{v}^1_{1,K1} & \overline{v}^2_{1,K1} & \cdots & \overline{v}^r_{1,K1} & 1 & 0 & 0 \\
\overline{v}^1_{1,K2} & \overline{v}^2_{1,K2} & \cdots & \overline{v}^r_{1,K2} & 1 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\overline{v}^1_{1,K_{N_e}} & \overline{v}^2_{1,K_{N_e}} & \cdots & \overline{v}^r_{1,K_{N_e}} & 1 & 0 & 0 \\
\overline{v}^1_{2,K1} & \overline{v}^2_{2,K1} & \cdots & \overline{v}^r_{2,K1} & 0 & 1 & 0 \\
\overline{v}^1_{2,K2} & \overline{v}^2_{2,K2} & \cdots & \overline{v}^r_{2,K2} & 0 & 1 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\overline{v}^1_{2,K_{N_e}} & \overline{v}^2_{2,K_{N_e}} & \cdots & \overline{v}^r_{2,K_{N_e}} & 0 & 1 & 0 \\
\overline{v}^1_{3,K1} & \overline{v}^2_{3,K1} & \cdots & \overline{v}^r_{3,K1} & 0 & 0 & 1 \\
\overline{v}^1_{3,K2} & \overline{v}^2_{3,K2} & \cdots & \overline{v}^r_{3,K2} & 0 & 0 & 1 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\overline{v}^1_{3,K_{N_e}} & \overline{v}^2_{3,K_{N_e}} & \cdots & \overline{v}^r_{3,K_{N_e}} & 0 & 0 & 1
\end{bmatrix}$$

Remark

The matrix $\overline{\mathbb{V}}^{r^\star}$ *is not orthogonal. We employ a QR factorization on the matrix* $\overline{\mathbb{V}}^{r^\star}$ *so that* $\overline{\mathbb{V}}^{r^\star} = \mathbb{Q} \times \widetilde{\mathbb{R}}$ *where* $\mathbb{Q} \in \mathbb{R}^{N_s \times N_e, r^\star}$ *is orthogonal, and* $\widetilde{\mathbb{R}} \in \mathbb{R}^{r^\star, r^\star}$ *is upper triangular.*

Remark

The matrix $\overline{\mathbb{V}}^{r^\star}$ is not orthogonal. We employ a QR factorization on the matrix $\overline{\mathbb{V}}^{r^\star}$ so that $\overline{\mathbb{V}}^{r^\star} = \mathbb{Q} \times \widetilde{\mathbb{R}}$ where $\mathbb{Q} \in \mathbb{R}^{N_s \times N_e, r^\star}$ is orthogonal, and $\widetilde{\mathbb{R}} \in \mathbb{R}^{r^\star, r^\star}$ is upper triangular.

For each $\mu \in \mathcal{P}$, at each time step $n = 1 \cdots N_{\mathrm{t}}$, we define a "temporary reduced solution" denoted by $\overline{\boldsymbol{Z}}^n_\mu \in \mathbb{R}^{N_s \times N_e}$. It is expressed in the basis $(\boldsymbol{Q}^1, \cdots, \boldsymbol{Q}^{r^\star})$ as

$$\overline{\boldsymbol{Z}}^n_\mu := \sum_{k=1}^{r^\star} \overline{c}^{k,n}_\mu \boldsymbol{Q}^k \quad \text{and} \quad \overline{\boldsymbol{Z}}^0_\mu := \Pi_{\mathrm{Span}(\boldsymbol{Q}^1, \cdots, \boldsymbol{Q}^{r^\star})} \boldsymbol{Z}^0_\mu \quad \text{where} \quad z^0_{\mu,i,K} := \ln(u^0_{\mu,i,K}).$$

Remark

The matrix $\overline{\mathbb{V}}^{r^\star}$ is not orthogonal. We employ a QR factorization on the matrix $\overline{\mathbb{V}}^{r^\star}$ so that $\overline{\mathbb{V}}^{r^\star} = \mathbb{Q} \times \widetilde{\mathbb{R}}$ where $\mathbb{Q} \in \mathbb{R}^{N_s \times N_e, r^\star}$ is orthogonal, and $\widetilde{\mathbb{R}} \in \mathbb{R}^{r^\star, r^\star}$ is upper triangular.

For each $\mu \in \mathcal{P}$, at each time step $n = 1 \cdots N_t$, we define a "temporary reduced solution" denoted by $\overline{\boldsymbol{Z}}_\mu^n \in \mathbb{R}^{N_s \times N_e}$. It is expressed in the basis $(\boldsymbol{Q}^1, \cdots, \boldsymbol{Q}^{r^\star})$ as

$$\overline{\boldsymbol{Z}}_\mu^n := \sum_{k=1}^{r^\star} \overline{c}_\mu^{k,n} \boldsymbol{Q}^k \quad \text{and} \quad \overline{\boldsymbol{Z}}_\mu^0 := \Pi_{\mathrm{Span}(\boldsymbol{Q}^1, \cdots, \boldsymbol{Q}^{r^\star})} \boldsymbol{Z}_\mu^0 \quad \text{where} \quad z_{\mu,i,K}^0 := \ln(u_{\mu,i,K}^0).$$

Definition of the coefficient $\overline{\boldsymbol{c}}_\mu^{k,n}$

Solve the nonlinear problem $\overline{H}_l(\overline{\boldsymbol{c}}_\mu^n) = 0$ with $\overline{H}_l(\overline{\boldsymbol{c}}_\mu^n) := \left\langle \boldsymbol{Q}^l, G^n(\overline{\boldsymbol{U}}_\mu^n) \right\rangle \quad \forall 1 \le l \le r^\star$.

How can we construct a structure preserving reduced model ?

Safe reduced solution

$$\overline{\boldsymbol{U}}^n_{\mu} := (\overline{u}^n_{\mu,i,K})_{i\in[1,N_s],K\in\mathcal{T}_h} \quad \text{with} \quad \overline{u}^n_{\mu,i,K} := \exp(\overline{z}^n_{\mu,i,K}) / \sum_{j=1}^{N_s} \exp(\overline{z}^n_{\mu,j,K}).$$

Safe reduced solution

$$\overline{\boldsymbol{U}}^n_{\mu} := (\overline{u}^n_{\mu,i,K})_{i\in[1,N_s],K\in\mathcal{T}_h} \quad \text{with} \quad \overline{u}^n_{\mu,i,K} := \exp(\overline{z}^n_{\mu,i,K}) / \sum_{j=1}^{N_s} \exp(\overline{z}^n_{\mu,j,K}).$$

Structural properties of the reduced solution

Safe reduced solution

$$\overline{\boldsymbol{U}}_{\mu}^{n} := (\overline{u}_{\mu,i,K}^{n})_{i\in[1,N_s],K\in\mathcal{T}_h} \quad \text{with} \quad \overline{u}_{\mu,i,K}^{n} := \exp(\overline{z}_{\mu,i,K}^{n}) / \sum_{j=1}^{N_s} \exp(\overline{z}_{\mu,j,K}^{n}).$$

Structural properties of the reduced solution

1. Positivity $\quad \overline{u}_{\mu,i,K}^{n} > 0 \quad \forall \mu \in \mathcal{P} \quad \forall K \in \mathcal{T}_h \quad \forall i \in [1, N_s] \quad \forall n \in [0, N_t].$

Safe reduced solution

$$\overline{\boldsymbol{U}}_{\mu}^{n} := (\overline{u}_{\mu,i,K}^{n})_{i\in[1,N_s],K\in\mathcal{T}_h} \quad \text{with} \quad \overline{u}_{\mu,i,K}^{n} := \exp(\overline{z}_{\mu,i,K}^{n}) / \sum_{j=1}^{N_s} \exp(\overline{z}_{\mu,j,K}^{n}).$$

Structural properties of the reduced solution

1. Positivity $\overline{u}_{\mu,i,K}^{n} > 0 \quad \forall \mu \in \mathcal{P} \quad \forall K \in \mathcal{T}_h \quad \forall i \in [1, N_s] \quad \forall n \in [0, N_t]$.
2. Volume filling constraint: $\sum_{i=1}^{N_s} \overline{u}_{\mu,i,K}^{n} = 1 \quad \forall \mu \in \mathcal{P} \quad \forall K \in \mathcal{T}_h, \quad \forall n \in [1, N_t]$.

Safe reduced solution

$$\overline{\boldsymbol{U}}^n_{\mu} := (\overline{u}^n_{\mu,i,K})_{i\in[1,N_s],K\in\mathcal{T}_h} \quad \text{with} \quad \overline{u}^n_{\mu,i,K} := \exp(\overline{z}^n_{\mu,i,K}) / \sum_{j=1}^{N_s} \exp(\overline{z}^n_{\mu,j,K}).$$

Structural properties of the reduced solution

1. Positivity $\quad \overline{u}^n_{\mu,i,K} > 0 \quad \forall \mu \in \mathcal{P} \quad \forall K \in \mathcal{T}_h \quad \forall i \in [1, N_s] \quad \forall n \in [0, N_{\mathrm{t}}]$.
2. Volume filling constraint: $\sum_{i=1}^{N_s} \overline{u}^n_{\mu,i,K} = 1 \quad \forall \mu \in \mathcal{P} \quad \forall K \in \mathcal{T}_h, \quad \forall n \in [1, N_{\mathrm{t}}]$.
3. mass conservation

$$\sum_{K\in\mathcal{T}_h} |K| \overline{u}^n_{\mu,i,K} = \sum_{K\in\mathcal{T}_h} |K| \overline{u}^{n-1}_{\mu,i,K} = \int_{\Omega} \overline{u}^0_i(x)\,\mathrm{dx} \quad \forall i \in [1, N_s] \quad \forall n \in [1, N_{\mathrm{t}}] \,.$$

Safe reduced solution

$$\overline{\boldsymbol{U}}^n_\mu := (\overline{u}^n_{\mu,i,K})_{i\in[1,N_s],K\in\mathcal{T}_h} \quad \text{with} \quad \overline{u}^n_{\mu,i,K} := \exp(\overline{z}^n_{\mu,i,K}) / \sum_{j=1}^{N_s} \exp(\overline{z}^n_{\mu,j,K}).$$

Structural properties of the reduced solution

1. Positivity $\quad \overline{u}^n_{\mu,i,K} > 0 \quad \forall \mu \in \mathcal{P} \quad \forall K \in \mathcal{T}_h \quad \forall i \in [1, N_s] \quad \forall n \in [0, N_\mathrm{t}]$.
2. Volume filling constraint: $\sum_{i=1}^{N_s} \overline{u}^n_{\mu,i,K} = 1 \quad \forall \mu \in \mathcal{P} \quad \forall K \in \mathcal{T}_h, \quad \forall n \in [1, N_\mathrm{t}]$.
3. mass conservation

$$\sum_{K\in\mathcal{T}_h} |K| \overline{u}^n_{\mu,i,K} = \sum_{K\in\mathcal{T}_h} |K| \overline{u}^{n-1}_{\mu,i,K} = \int_\Omega \overline{u}^0_i(x)\,\mathrm{d}x \quad \forall i \in [1, N_s] \quad \forall n \in [1, N_\mathrm{t}]\,.$$

4. The discrete counterpart of the entropy decays along time

$$E_{\mathcal{T}_h}(\overline{\boldsymbol{U}}^n_\mu) - E_{\mathcal{T}_h}(\overline{\boldsymbol{U}}^{n-1}_\mu) + \Delta t_n \min_{i,j} a_{i,j} \sum_{\sigma\in\mathcal{E}_h^{\mathrm{int}}} \sum_{i=1}^{N_s} \tau_\sigma \overline{u}^n_{\mu,i\sigma} \left(D_{K\sigma}(\ln(\overline{\boldsymbol{u}}^n_{\boldsymbol{\mu},\boldsymbol{i}}))\right)^2 \le 0 \quad \forall n \in [1, N_\mathrm{t}]\,.$$

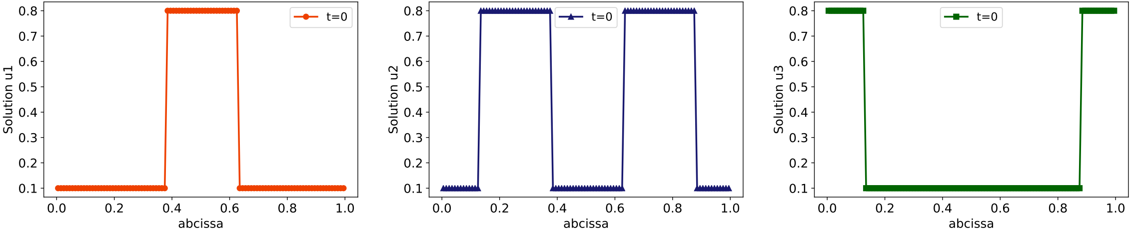
Outline

First test case

- We consider 3 species.
- Ω is a one dimensional domain consisting in a segment of length $L = 1m$.
- $\Delta x = 10^{-2}$.
- Final simulation time $T = 0.5$ s and $\Delta t = 5 \times 10^{-4}$ s.
- Compute $\mu = 20$ snapshots of solutions.

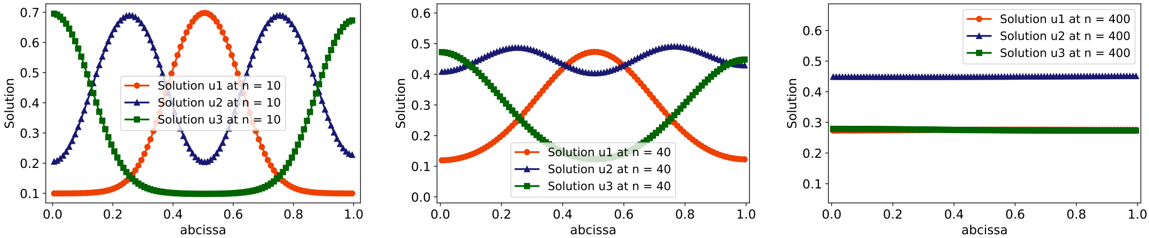
Initial condition

- discontinuous solution



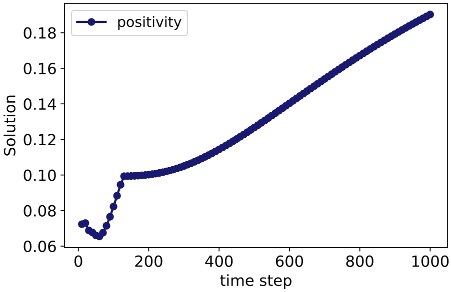
Shape of the solution

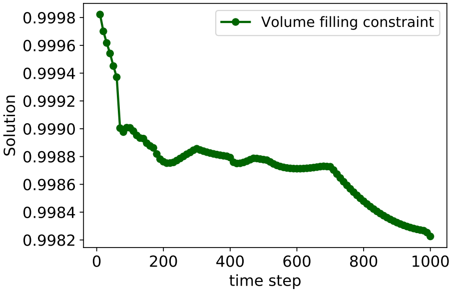
$$\mathbb{A}_{\mu} := \begin{pmatrix} 0 & 0.75 & 0.73 \\ 0.75 & 0 & 0.84 \\ 0.73 & 0.84 & 0 \end{pmatrix}$$



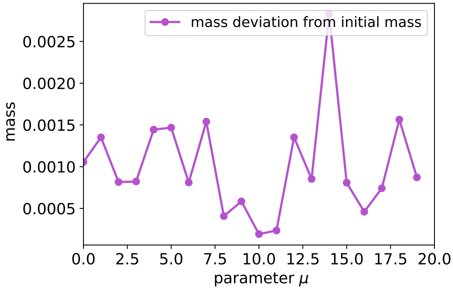
Typical behavior of a cross-diffusion system

Structural properties of the solution





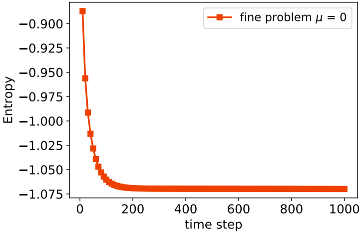
$$\mathcal{P}_U(t_n) := \inf_{\mu \in \mathcal{P}^{\text{off}}} \inf_{K \in \mathcal{T}_h} U^n_{\mu,K} \qquad \mathcal{S}_U(t_n) := \inf_{\mu \in \mathcal{P}^{\text{off}}} \inf_{K \in \mathcal{T}_h} \sum_{i=1}^{N_s} U^n_{\mu,i,K}$$

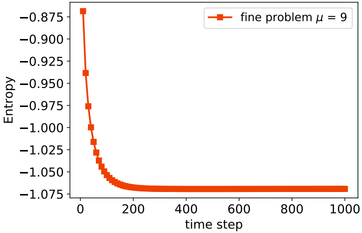


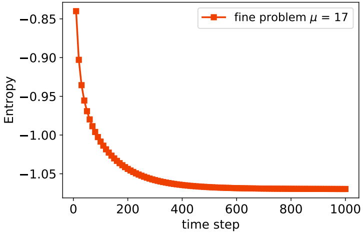
$$\mathcal{E}_U(\mu) := \max_{i \in [\![1, N_s]\!]} \max_{n \in [\![1, N_t]\!]} \left| \sum_{K \in \mathcal{T}_h} |K| U^n_{\mu, i, K} - \int_\Omega u_i{}^0(x)\,\mathrm{dx} \right|$$

Structural properties of the Solution

$$\mathbb{A}_{\mu_0} := \begin{pmatrix} 0 & 0.75 & 0.73 \\ 0.75 & 0 & 0.84 \\ 0.73 & 0.84 & 0 \end{pmatrix} \mathbb{A}_{\mu_9} = \begin{pmatrix} 0 & 0.93 & 0.71 \\ 0.93 & 0 & 0.44 \\ 0.71 & 0.44 & 0 \end{pmatrix} \quad \mathbb{A}_{\mu_{17}} = \begin{pmatrix} 0 & 0.37 & 0.004 \\ 0.37 & 0 & 0.72 \\ 0.004 & 0.72 & 0 \end{pmatrix}$$

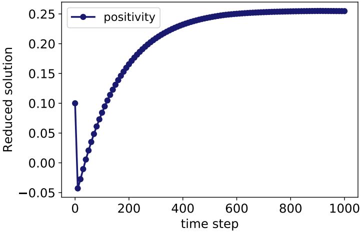




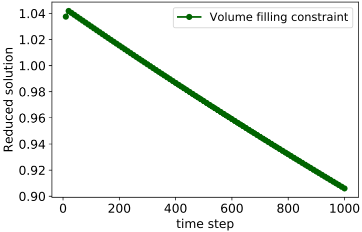


First POD reduced model

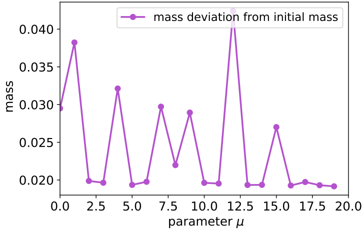
Violation of the structural properties of the solution



$r = 2$

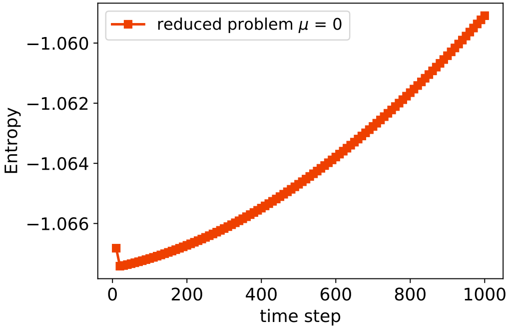


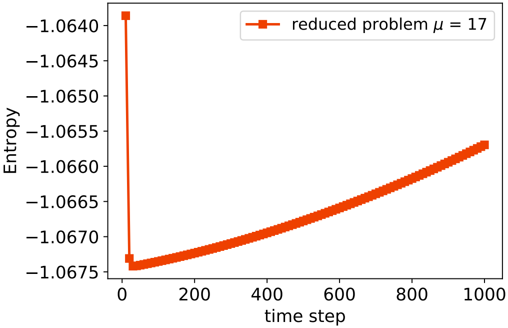
$r = 1$



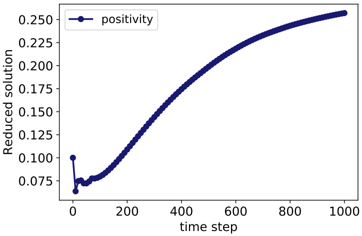
$r = 1$

$$\mathbb{A}_0 = \begin{pmatrix} 0 & 0.93 & 0.71 \\ 0.93 & 0 & 0.44 \\ 0.71 & 0.44 & 0 \end{pmatrix} \qquad \mathbb{A}_{17} = \begin{pmatrix} 0 & 0.37 & 0.004 \\ 0.37 & 0 & 0.72 \\ 0.004 & 0.72 & 0 \end{pmatrix}$$

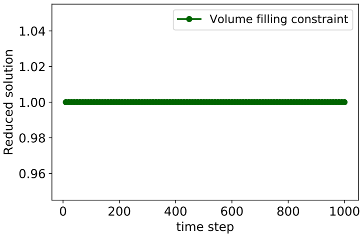




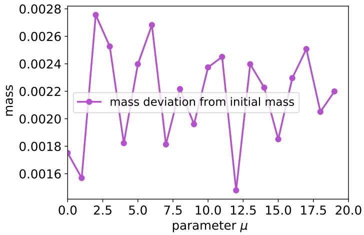
Second POD reduced model



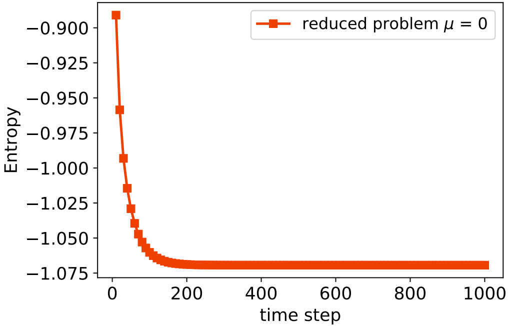
$r = 2$

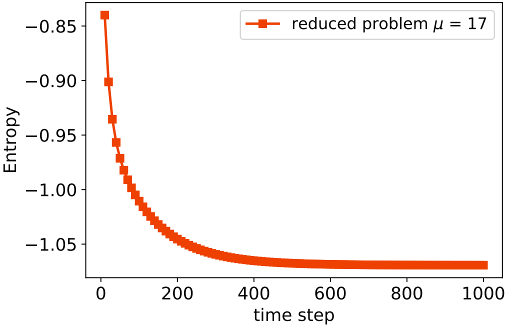


$r = 1$



$r = 1$





The entropy decreases with respect to time.

Second test case

- We consider the PVD process : 4 species.
- Ω is a one dimensional domain consisting in a segment of length $L = 1m$.
- $\Delta x = 10^{-2}$.
- Final simulation time $T = 0.5$ s and $\Delta t = 2.5 \times 10^{-4}$ s.
- Compute $\mu = 20$ snapshots of solutions.

Initial condition

We take

$$w_1^0(x) = e^{-25(x-0.5)^2}, \quad w_2^0(x) = x^2 + \varepsilon, \quad w_3(x) = 1 - e^{-25(x-0.5)^2}, \quad w_4(x) = |\sin(\pi x)|$$

where $\varepsilon = 10^{-6}$.

To satisfy the volume filling constraint property we use a renormalization

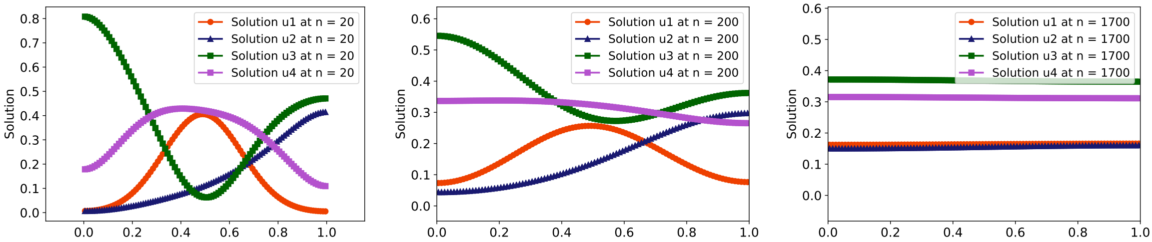
$$u_i{}^0(x_j) = \frac{w_i^0(x_j)}{\sum_{l=1}^{N_s} w_l^0(x_j)}$$

where x_j, $j \in [1, N_e]$ are the cell centers of the mesh.

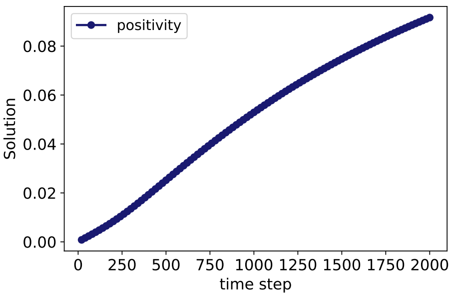
Fine solution

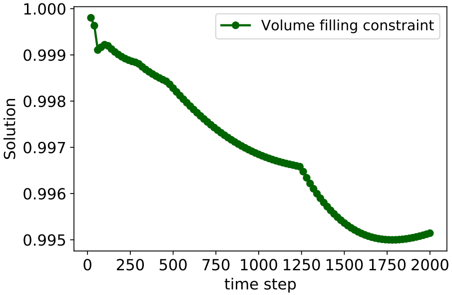
Cross-diffusion matrix

$$\mathbb{A}_{17} = \begin{pmatrix} 0 & 0.64 & 0.31 & 0.53 \\ 0.64 & 0 & 0.99 & 0.84 \\ 0.32 & 0.99 & 0 & 0.99 \\ 0.53 & 0.84 & 0.99 & 0 \end{pmatrix}$$



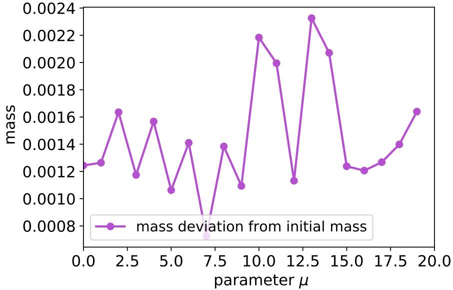
Properties of the solution





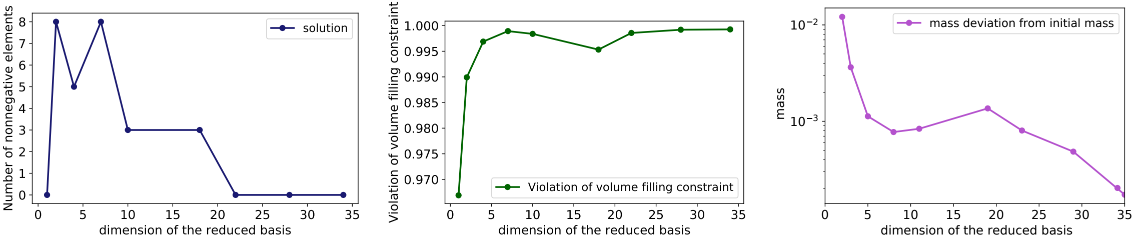
$$\mathcal{P}_{\overline{U}}(t_n) := \inf_{\mu \in \mathcal{P}^{\text{off}}} \inf_{K \in \mathcal{T}_h} \overline{U}^n_{\mu,K}$$

$$\mathcal{S}_{\overline{U}}(t_n) := \inf_{\mu \in \mathcal{P}^{\text{off}}} \inf_{K \in \mathcal{T}_h} \sum_{i=1}^{N_s} \overline{U}^n_{\mu,i,K}$$



$$\mathcal{E}_{\overline{U}}(\mu) := \max_{i\in[\![1,N_s]\!]} \max_{n\in[\![1,N_t]\!]} \left| \sum_{K\in\mathcal{T}_h} |K| \overline{U}^n_{\mu,i,K} - \int_\Omega \overline{u}^0_i(x)\,\mathrm{dx} \right|$$

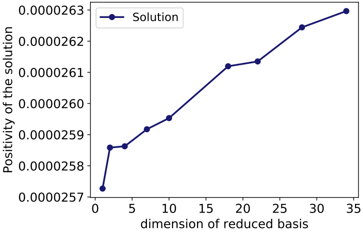
First POD reduced model

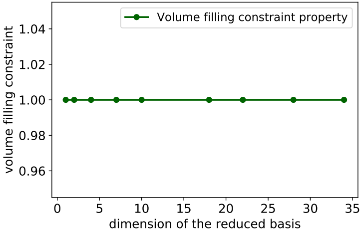


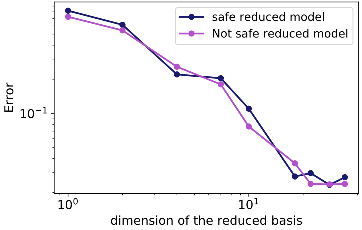
$$\inf_{n\in[\![1,N_t]\!]} \inf_{\mu\in\mathcal{P}^{\mathrm{off}}} \inf_{K\in\mathcal{T}_h} \sum_{i=1}^{N_s} \widetilde{U}^n_{\mu,i,K} \qquad \max_{\mu\in\mathcal{P}^{\mathrm{off}}} \max_{i\in[\![1,N_s]\!]} \max_{n\in[\![1,N_t]\!]} \left| \sum_{K\in\mathcal{T}_h} |K| \overline{U}^n_{\mu,i,K} - \int_\Omega \overline{u}_i^0(x)\,\mathrm{d}x \right|$$

Violation of the physical properties.

Safe POD reduced model







$$\inf_{n\in[\![1,N_t]\!]} \inf_{\mu\in\mathcal{P}^{\text{off}}} \inf_{K\in\mathcal{T}_h} \overline{U}^n_{\mu,K} \qquad \inf_{n\in[\![1,N_t]\!]} \inf_{\mu\in\mathcal{P}^{\text{off}}} \inf_{K\in\mathcal{T}_h} \sum_{i=1}^{N_s} \overline{U}^n_{\mu,i,K}$$

$$\max_{i\in[\![1,N_s]\!]} \left\| u^i_\mu - u^{i,\text{red}}_\mu \right\|_{L^\infty(\mathcal{P}^{\text{off}},L^2(\Omega),L^\infty([0,T]))} := \max_{\mu\in\mathcal{P}^{\text{off}}} \max_{n\in[\![1,N_t]\!]} \left(\sum_{K\in\mathcal{T}_h}\left|U^n_{\mu,i,K} - U^{n,\text{red}}_{\mu,i,K}\right|^2\right)^{\frac{1}{2}}$$

Outline

Conclusion and perspectives

Conclusion

- We constructed an efficient reduced model preserving the physical properties of the cross-diffusion system.

Perspectives

- Construct a reduced basis method with a posteriori estimators for high accuracy.
- EIM algorithm to reduce the computational time.