A posteriori error estimates for variational inequalities: application to a two-phase flow in porous media

LMAC seminar, UTC

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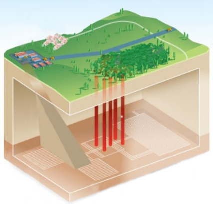
Outline

Motivation

Study a simplified mathematical model for the storage of radioactive waste

$$\begin{cases} \partial_t l_{\mathrm{w}}(S^{\mathrm{l}}) + \nabla \cdot \Phi_{\mathrm{w}}(S^{\mathrm{l}}, P^{\mathrm{l}}, \chi_{\mathrm{h}}^{\mathrm{l}}) = Q_{\mathrm{w}} \\ \partial_t l_{\mathrm{h}}(S^{\mathrm{l}}, \chi_{\mathrm{h}}^{\mathrm{l}}) + \nabla \cdot \Phi_{\mathrm{h}}(S^{\mathrm{l}}, P^{\mathrm{l}}, \chi_{\mathrm{h}}^{\mathrm{l}}) = Q_{\mathrm{h}} \\ \mathcal{K}(S^{\mathrm{l}}) \geq 0, \ \mathcal{G}(S^{\mathrm{l}}, P^{\mathrm{l}}, \chi_{\mathrm{h}}^{\mathrm{l}}) \geq 0, \ \mathcal{K}(S^{\mathrm{l}}) \cdot \mathcal{G}(S^{\mathrm{l}}, P^{\mathrm{l}}, \chi_{\mathrm{h}}^{\mathrm{l}}) = 0 \end{cases}$$



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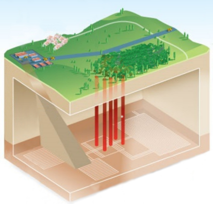

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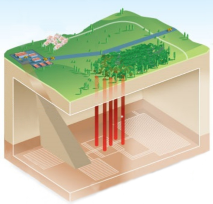

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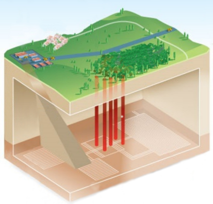

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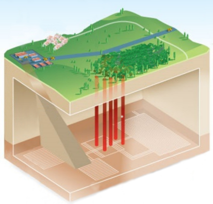

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We study 2 problems of increasing difficulty.

Motivation

Consider the system of PDEs with nonlinear complementarity constraints:

$$\partial_t \left(\varphi(\boldsymbol{u})\right) + \mathcal{A}(\boldsymbol{u}) = \mathcal{F}$$
$$\mathcal{K}(\boldsymbol{u}) \geq 0 \quad \mathcal{G}(\boldsymbol{u}) \geq 0 \quad \mathcal{K}(\boldsymbol{u}) \cdot \mathcal{G}(\boldsymbol{u}) = 0$$

This model is used in various physical phenomena: economy, fluid mechanics, elasticity, multiphase flow.

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Numerical resolution:

- Discretization: Finite elements /finite volumes + backward Euler scheme in time

$$\frac{\varphi(\boldsymbol{u}_h^n) - \varphi(\boldsymbol{u}_h^{n-1})}{t^n - t^{n-1}} + \mathcal{A}(\boldsymbol{u}_h^n) = \mathcal{F}_h^{n-1}$$
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- Nonlinear resolution: inexact semismooth Newton algorithm

$$\mathbb{A}^{n,k-1} \boldsymbol{U}_h^{n,k,i} + \boldsymbol{R}_h^{n,k,i} = \boldsymbol{F}^{n,k-1}$$

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A posteriori error estimate:

$$\left|\left|\left| \boldsymbol{u} - \boldsymbol{u}_h^{n,k,i} \right|\right|\right| \leq \eta(\boldsymbol{u}_h^{n,k,i}) \quad \text{where} \quad |||\cdot||| \quad \text{is some norm}$$

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Three components of the error:

- discretization error: numerical scheme (finite elements, finite volumes...)
- linearization error: semismooth Newton method (k)
- algebraic error: iterative algebraic solver (i)

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Questions:

Can we estimate each component of the error? **yes!**

- $\left|\!\left|\!\left| \boldsymbol{u} - \boldsymbol{u}_h^{n,k,i} \right|\!\right|\!\right| \leq \eta_{\mathrm{disc}}^{n,k,i} + \eta_{\mathrm{lin}}^{n,k,i} + \eta_{\mathrm{alg}}^{n,k,i}$

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Can we reduce the number of iterations? **yes!**

- Adaptive stopping criterion: semismooth linearization: $\eta_{\mathrm{lin}}^{n,k,i} \leq \gamma_{\mathrm{lin}} \eta_{\mathrm{disc}}^{n,k,i}$
- Adaptive stopping criterion: algebraic: $\eta_{\mathrm{alg}}^{n,k,i} \leq \gamma_{\mathrm{alg}} \left\{ \eta_{\mathrm{disc}}^{n,k,i}, \eta_{\mathrm{lin}}^{n,k,i} \right\}$

Main contribution

Stationary linear variational inequality

$$\text{Find} \quad \boldsymbol{u} \in \mathcal{K}_g \quad a(\boldsymbol{u}, \boldsymbol{v} - \boldsymbol{u}) \geq (\boldsymbol{f}, \boldsymbol{v} - \boldsymbol{u}) \quad \forall \boldsymbol{v} \in \mathcal{K}_g$$

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Two-phase compositional flow in porous media

$$\text{Find} \quad S^{\mathrm{l}}, P^{\mathrm{l}}, \chi_{\mathrm{h}}^{\mathrm{l}} \quad \begin{cases} \partial_t l_{\mathrm{w}}(S^{\mathrm{l}}) + \nabla \cdot \Phi_{\mathrm{w}}(S^{\mathrm{l}}, P^{\mathrm{l}}, \chi_{\mathrm{h}}^{\mathrm{l}}) = Q_{\mathrm{w}}, \\ \partial_t l_{\mathrm{h}}(S^{\mathrm{l}}, \chi_{\mathrm{h}}^{\mathrm{l}}) + \nabla \cdot \Phi_{\mathrm{h}}(S^{\mathrm{l}}, P^{\mathrm{l}}, \chi_{\mathrm{h}}^{\mathrm{l}}) = Q_{\mathrm{h}}, \\ \mathcal{K}(S^{\mathrm{l}}) \geq 0, \ \mathcal{G}(S^{\mathrm{l}}, P^{\mathrm{l}}, \chi_{\mathrm{h}}^{\mathrm{l}}) \geq 0, \ \mathcal{K}(S^{\mathrm{l}}) \cdot \mathcal{G}(S^{\mathrm{l}}, P^{\mathrm{l}}, \chi_{\mathrm{h}}^{\mathrm{l}}) = 0 \end{cases}$$

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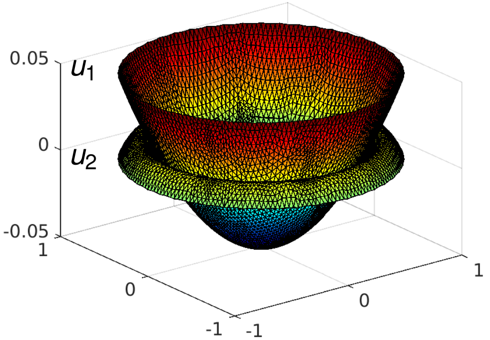
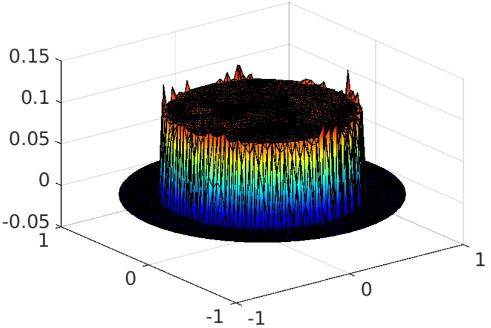
Outline

Model problem and settings: contact between two membranes

Find u_1, u_2, λ such that
$$\begin{cases} -\mu_1 \Delta u_1 - \lambda = f_1 & \text{in} \quad \Omega, \\ -\mu_2 \Delta u_2 + \lambda = f_2 & \text{in} \quad \Omega, \\ (u_1 - u_2)\lambda = 0, \quad u_1 - u_2 \geq 0, \quad \lambda \geq 0 & \text{in} \quad \Omega, \\ u_1 = g > 0 & \text{on} \quad \partial\Omega, \\ u_2 = 0 & \text{on} \quad \partial\Omega. \end{cases}$$





Continuous problem

- $H^1_g(\Omega) = \{u \in H^1(\Omega),\ u = g \text{ on } \partial\Omega\}$ $\quad \Lambda = \{\chi \in L^2(\Omega),\ \chi \geq 0 \text{ on } \Omega\}$

Saddle point type weak formulation: For $(f_1, f_2) \in \left[L^2(\Omega)\right]^2$ and $g > 0$ find $(u_1, u_2, \lambda) \in H^1_g(\Omega) \times H^1_0(\Omega) \times \Lambda$ such that

$$\begin{cases} \sum_{\alpha=1}^{2} \mu_\alpha (\nabla u_\alpha, \nabla v_\alpha)_\Omega - (\lambda, v_1 - v_2)_\Omega = \sum_{\alpha=1}^{2} (f_\alpha, v_\alpha)_\Omega \quad \forall (v_1, v_2) \in \left[H^1_0(\Omega)\right]^2 \\ (\chi - \lambda, u_1 - u_2)_\Omega \geq 0 \quad \forall \chi \in \Lambda \end{cases} \quad (\mathrm{S})$$

equivalent to

Variational inequality:

- $\mathcal{K}_g = \{(v_1, v_2) \in H^1_g(\Omega) \times H^1_0(\Omega),\ v_1 - v_2 \geq 0 \text{ on } \Omega\}$

$$\sum_{\alpha=1}^{2} \mu_\alpha (\nabla u_\alpha, \nabla (v_\alpha - u_\alpha))_\Omega \geq \sum_{\alpha=1}^{2} (f_\alpha, v_\alpha - u_\alpha)_\Omega \quad \forall \mathbf{v} = (v_1, v_2) \in \mathcal{K}_g \qquad (\mathrm{R})$$

Discretization by finite elements

For any $p \geq 1$

Spaces for the discretization:

$$X_{gh}^{p} = \left\{ v_h \in \mathcal{C}^0(\overline{\Omega}), v_{h|K} \in \mathbb{P}_p(K),\ \forall K \in \mathcal{T}_h,\ \ v_h = g \text{ on } \partial\Omega \right\}$$

$$X_{0h}^{p} = \left\{ v_h \in \mathcal{C}^0(\overline{\Omega});\ v_h|_K \in \mathbb{P}_p(K),\quad \forall K \in \mathcal{T}_h \right\} \cap H_0^1(\Omega)$$

$$\mathcal{K}_{gh}^{p} = \left\{ (v_{1h}, v_{2h}) \in X_{gh}^{p} \times X_{0h}^{p},\ v_{1h}(\boldsymbol{x}_l) - v_{2h}(\boldsymbol{x}_l) \geq 0\ \ \forall \boldsymbol{x}_l \in \mathcal{V}_{\mathrm{d}}^{p} \right\} \not\subset \mathcal{K}_g \quad \forall p \geq 2$$

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Discrete variational inequality: find $\boldsymbol{u}_h = (u_{1h}, u_{2h}) \in \mathcal{K}_{gh}^{p}$ such that

$$\sum_{\alpha=1}^{2} \mu_\alpha \left(\nabla u_{\alpha h}, \nabla \left(v_{\alpha h} - u_{\alpha h}\right)\right)_\Omega \geq \sum_{\alpha=1}^{2} \left(f_\alpha, v_{\alpha h} - u_{\alpha h}\right)_\Omega \quad \forall \boldsymbol{v}_h = (v_{1h}, v_{2h}) \in \ \mathcal{K}_g \quad \text{(DR)}$$

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Resolution techniques: Projected Newton methods (Bertsekas 1982), Active set Newton method (Kanzow 1999), Primal-dual active set strategy (Hintermüller 2002).

Saddle point formulation

Characterization of the discrete lagrange multiplier: Define λ_{1h} and λ_{2h} in X_h^p by

$$\begin{cases} \langle \lambda_{1h}, z_{1h} \rangle_h & := \mu_1 \left(\nabla u_{1h}, \nabla z_{1h} \right)_\Omega - (f_1, z_{1h})_\Omega & \forall z_{1h} \in X_{0h}^p, \\ \langle \lambda_{2h}, z_{2h} \rangle_h & := -\mu_2 \left(\nabla u_{2h}, \nabla z_{2h} \right)_\Omega + (f_2, z_{2h})_\Omega & \forall z_{2h} \in X_{0h}^p, \\ \langle \lambda_{1h}, \psi_{h,\boldsymbol{x}_l} \rangle_h & := \langle \lambda_{2h}, \psi_{h,\boldsymbol{x}_l} \rangle_h = 0 & \forall \boldsymbol{x}_l \in \mathcal{V}_{\mathrm{d}}^{p,\mathrm{ext}}, \end{cases}$$

where for all $(w_h, v_h) \in X_h^p \times X_h^p$

$$\langle w_h, v_h \rangle_h := \sum_{\boldsymbol{a} \in \mathcal{V}_h} w_h(\boldsymbol{a}) v_h(\boldsymbol{a}) M_{\boldsymbol{a}} \quad \text{if} \quad p = 1 \quad \text{and} \quad \langle w_h, v_h \rangle_h := (w_h, v_h)_\Omega \quad \text{if} \quad p \geq 2$$

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$p = 1$: $\Lambda_h^1 := \left\{ v_h \in X_{0h}^1 \; v_h(\boldsymbol{a}) \geq 0 \; \forall \boldsymbol{a} \in \mathcal{V}_{\mathrm{d}}^{p,\mathrm{int}} \right\} \subset \boldsymbol{\Lambda}$ Ben Belgacem, Bernardi, Blouza, and Vohralík (2008)

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where for all $(w_h, v_h) \in X_h^p \times X_h^p$

$$\langle w_h, v_h \rangle_h := \sum_{\mathbf{a} \in \mathcal{V}_h} w_h(\mathbf{a}) v_h(\mathbf{a}) M_{\mathbf{a}} \quad \text{if} \quad p = 1 \quad \text{and} \quad \langle w_h, v_h \rangle_h := (w_h, v_h)_\Omega \quad \text{if} \quad p \geq 2$$

$p = 1$: $\Lambda_h^1 := \left\{ v_h \in X_{0h}^1 \; v_h(\mathbf{a}) \geq 0 \; \forall \mathbf{a} \in \mathcal{V}_{\mathrm{d}}^{p,\mathrm{int}} \right\} \subset \mathbf{\Lambda}$ Ben Belgacem, Bernardi, Blouza, and Vohralík (2008)

$p \geq 2$ **(new):** $\Lambda_h^p := \left\{ v_h \in X_h^p \; \langle v_h, \psi_{h,\mathbf{x}_l} \rangle_h \geq 0 \; \forall \mathbf{x}_l \in \mathcal{V}_{\mathrm{d}}^{p,\mathrm{int}} \; \langle v_h, \psi_{h,\mathbf{x}_l} \rangle_h = 0 \; \forall \mathbf{x}_l \in \mathcal{V}_{\mathrm{d}}^{p,\mathrm{ext}} \right\} \not\subset \mathbf{\Lambda}$

Discrete saddle point type formulation Find $(u_{1h}, u_{2h}, \lambda_h) \in X^p_{gh} \times X^p_{0h} \times \Lambda^p_h$ such that $\forall (z_{1h}, z_{2h}) \in [X^p_{0h}]^2$

$$\sum_{\alpha=1}^{2} \mu_\alpha (\nabla u_{\alpha h}, \nabla z_{\alpha h})_\Omega - \langle \lambda_h, z_{1h} - z_{2h} \rangle_h = \sum_{\alpha=1}^{2} (f_\alpha, z_{\alpha h})_\Omega \quad \text{(DS)},$$

$$\langle \chi_h - \lambda_h, u_{1h} - u_{2h} \rangle_h \geq 0 \quad \forall \chi_h \in \Lambda^p_h.$$

Discrete saddle point type formulation Find $(u_{1h}, u_{2h}, \lambda_h) \in X_{gh}^p \times X_{0h}^p \times \Lambda_h^p$ such that $\forall (z_{1h}, z_{2h}) \in [X_{0h}^p]^2$

$$\sum_{\alpha=1}^{2} \mu_\alpha \left(\nabla u_{\alpha h}, \nabla z_{\alpha h}\right)_\Omega - \langle \lambda_h, z_{1h} - z_{2h} \rangle_h = \sum_{\alpha=1}^{2} (f_\alpha, z_{\alpha h})_\Omega \qquad \text{(DS)},$$

$$\langle \chi_h - \lambda_h, u_{1h} - u_{2h} \rangle_h \geq 0 \quad \forall \chi_h \in \Lambda_h^p.$$

Lemma

(DR) ***equivalent to*** (DS)

Discrete saddle point type formulation Find $(u_{1h}, u_{2h}, \lambda_h) \in X^p_{gh} \times X^p_{0h} \times \Lambda^p_h$ such that $\forall (z_{1h}, z_{2h}) \in [X^p_{0h}]^2$

$$\sum_{\alpha=1}^{2} \mu_\alpha (\nabla u_{\alpha h}, \nabla z_{\alpha h})_\Omega - \langle \lambda_h, z_{1h} - z_{2h} \rangle_h = \sum_{\alpha=1}^{2} (f_\alpha, z_{\alpha h})_\Omega \quad \text{(DS)},$$
$$\langle \chi_h - \lambda_h, u_{1h} - u_{2h} \rangle_h \geq 0 \quad \forall \chi_h \in \Lambda^p_h.$$

Lemma

(DR) ***equivalent to*** (DS)

- $p = 1$: Ben Belgacem, Bernardi, Blouza, and Vohralík (2008)

Discrete saddle point type formulation Find $(u_{1h}, u_{2h}, \lambda_h) \in X^p_{gh} \times X^p_{0h} \times \Lambda^p_h$ such that $\forall (z_{1h}, z_{2h}) \in [X^p_{0h}]^2$

$$\sum_{\alpha=1}^{2} \mu_\alpha (\nabla u_{\alpha h}, \nabla z_{\alpha h})_\Omega - \langle \lambda_h, z_{1h} - z_{2h} \rangle_h = \sum_{\alpha=1}^{2} (f_\alpha, z_{\alpha h})_\Omega \quad \text{(DS)},$$

$$\langle \chi_h - \lambda_h, u_{1h} - u_{2h} \rangle_h \geq 0 \quad \forall \chi_h \in \Lambda^p_h.$$

Lemma

(DR) ***equivalent to*** (DS)

- $p = 1$: Ben Belgacem, Bernardi, Blouza, and Vohralík (2008)
- $p \geq 2$: New $\rightarrow$ Construction of the basis $(\Theta_{h,\boldsymbol{x}_l})_{1 \leq l \leq \mathcal{N}^p_{\mathrm{d}}}$ of X^p_h, dual to $(\psi_{h,\boldsymbol{x}_l})_{1 \leq l \leq \mathcal{N}^p_{\mathrm{d}}}$.

$$\left\langle \Theta_{h,\boldsymbol{x}_l}, \psi_{h,\boldsymbol{x}_{l'}} \right\rangle_h = \delta_l^{l'}$$

$(u_{1h}, u_{2h}, \lambda_h)$ solution of (DS) $\Rightarrow \langle \chi_h, u_{1h} - u_{2h} \rangle_h \geq 0 \ \forall \chi_h \in \Lambda^p_h$. Take $\chi_h = \Theta_{h,\boldsymbol{x}_l}$, $\boldsymbol{x}_l \in \mathcal{V}^{p,\mathrm{int}}_{\mathrm{d}}$

$$\sum_{\boldsymbol{x}'_l \in \mathcal{V}^p_{\mathrm{d}}} (u_{1h} - u_{2h})(\boldsymbol{x}'_l) \left\langle \Theta_{h,\boldsymbol{x}_l}, \psi_{h,\boldsymbol{x}_{l'}} \right\rangle_h = (u_{1h} - u_{2h})(\boldsymbol{x}'_l) \geq 0 \Rightarrow \boldsymbol{u}_h \in \mathcal{K}^p_{gh}$$

Discrete complementarity problems

$$\sum_{\alpha=1}^{2} \mu_\alpha (\nabla u_{\alpha h}, \nabla z_{\alpha h})_\Omega - \langle \lambda_h, z_{1h} - z_{2h} \rangle_h = \sum_{\alpha=1}^{2} (f_\alpha, z_{\alpha h})_\Omega \quad \forall (z_{1h}, z_{2h}) \in [X_{0h}^p]^2,$$

$$(u_{1h} - u_{2h})(\boldsymbol{x}_l) \geq 0 \ \forall \boldsymbol{x}_l \in \mathcal{V}_{\mathrm{d}}^{p,\mathrm{int}}, \ \left\langle \lambda_h, \psi_{h,\boldsymbol{x}_l} \right\rangle_h \geq 0 \ \forall \boldsymbol{x}_l \in \mathcal{V}_{\mathrm{d}}^{p,\mathrm{int}}, \ \langle \lambda_h, u_{1h} - u_{2h} \rangle_h = 0.$$

Expression of the discrete problem in the Lagrange basis and in the dual basis:

$p = 1$: Lagrange basis: $u_{1h} = u_{1h}^* + g$ where $u_{1h}^* \in X_{0h}^p$ and $g > 0$.

$$u_{1h} = \sum_{l=1}^{\mathcal{N}_h^{\mathrm{int}}} (\boldsymbol{X}_{1h})_l \, \psi_{h,\boldsymbol{x}_l} + g, \quad u_{2h} = \sum_{l=1}^{\mathcal{N}_h^{\mathrm{int}}} (\boldsymbol{X}_{2h})_l \, \psi_{h,\boldsymbol{x}_l} \in X_{0h}^p \quad \lambda_h = \sum_{l=1}^{\mathcal{N}_h^{\mathrm{int}}} (\boldsymbol{X}_{3h})_l \, \psi_{h,\boldsymbol{a}_l} \in X_{0h}^p,$$

$$\mathbb{E}_1 \boldsymbol{X}_h = \boldsymbol{F},$$

$$\boldsymbol{X}_{1h} + g\boldsymbol{1} - \boldsymbol{X}_{2h} \geq 0, \quad \boldsymbol{X}_{3h} \geq 0, \quad (\boldsymbol{X}_{1h} + g\boldsymbol{1} - \boldsymbol{X}_{2h}) \cdot \boldsymbol{X}_{3h} = 0.$$

$$\mathbb{E}_1 := \begin{bmatrix} \mu_1 \mathbb{S} & \boldsymbol{0} & -\mathbb{D} \\ \boldsymbol{0} & \mu_2 \mathbb{S} & +\mathbb{D} \end{bmatrix}$$

Discrete complementarity problems

$$\sum_{\alpha=1}^{2} \mu_\alpha (\nabla u_{\alpha h}, \nabla z_{\alpha h})_\Omega - \langle \lambda_h, z_{1h} - z_{2h} \rangle_h = \sum_{\alpha=1}^{2} (f_\alpha, z_{\alpha h})_\Omega \quad \forall (z_{1h}, z_{2h}) \in [X_{0h}^p]^2,$$

$$(u_{1h} - u_{2h})(\boldsymbol{x}_l) \geq 0 \ \forall \boldsymbol{x}_l \in \mathcal{V}_{\mathrm{d}}^{p,\mathrm{int}}, \ \left\langle \lambda_h, \psi_{h,\boldsymbol{x}_l} \right\rangle_h \geq 0 \ \forall \boldsymbol{x}_l \in \mathcal{V}_{\mathrm{d}}^{p,\mathrm{int}}, \ \langle \lambda_h, u_{1h} - u_{2h} \rangle_h = 0.$$

Expression of the discrete problem in the Lagrange basis and in the dual basis:

$p \geq 2$: Lagrange basis: The discrete lagrange multiplier λ_h is decomposed in the full space X_h^p as

$$\lambda_h = \sum_{l=1}^{\mathcal{N}_{\mathrm{d}}^p} \left(\widetilde{\boldsymbol{X}}_{3h} \right)_l \psi_{h,\boldsymbol{x}_l} \quad \text{with} \quad \widetilde{\boldsymbol{X}}_{3h} \in \mathbb{R}^{\mathcal{N}_{\mathrm{d}}^p}.$$

$$\widetilde{\mathbb{E}}_p \boldsymbol{X}_h = \boldsymbol{F}, \qquad \widetilde{\mathbb{E}}_p := \begin{bmatrix} \mu_1 \mathbb{S} & \boldsymbol{0} & -\widehat{\mathbb{M}} \\ \boldsymbol{0} & \mu_2 \mathbb{S} & +\widehat{\mathbb{M}} \end{bmatrix}$$

$$\boldsymbol{X}_{1h} + g\boldsymbol{1} - \boldsymbol{X}_{2h} \geq 0, \quad \widehat{\mathbb{M}} \widetilde{\boldsymbol{X}}_{3h} \geq 0, \quad (\boldsymbol{X}_{1h} + g\boldsymbol{1} - \boldsymbol{X}_{2h}) \cdot \widehat{\mathbb{M}} \widetilde{\boldsymbol{X}}_{3h} = 0.$$

Discrete complementarity problems

$$\sum_{\alpha=1}^{2} \mu_\alpha (\nabla u_{\alpha h}, \nabla z_{\alpha h})_\Omega - \langle \lambda_h, z_{1h} - z_{2h} \rangle_h = \sum_{\alpha=1}^{2} (f_\alpha, z_{\alpha h})_\Omega \quad \forall (z_{1h}, z_{2h}) \in [X_{0h}^p]^2,$$

$$(u_{1h} - u_{2h})(\boldsymbol{x}_l) \geq 0 \ \forall \boldsymbol{x}_l \in \mathcal{V}_{\mathrm{d}}^{p,\mathrm{int}}, \ \langle \lambda_h, \psi_{h,\boldsymbol{x}_l} \rangle_h \geq 0 \ \forall \boldsymbol{x}_l \in \mathcal{V}_{\mathrm{d}}^{p,\mathrm{int}}, \ \langle \lambda_h, u_{1h} - u_{2h} \rangle_h = 0.$$

Expression of the discrete problem in the Lagrange basis and in the dual basis:

$p \geq 2$: **Dual basis:** The discrete Lagrange multiplier λ_h is decomposed in the basis $\Theta_{h,\boldsymbol{x}_l}$ as

$$\lambda_h = \sum_{l=1}^{\mathcal{N}_{\mathrm{d}}^{p,\mathrm{int}}} (\boldsymbol{X}_{3h})_l \, \Theta_{h,\boldsymbol{x}_l}, \quad \text{with} \quad \boldsymbol{X}_{3h} \in \mathbb{R}^{\mathcal{N}_{\mathrm{d}}^{p,\mathrm{int}}}.$$

$$\mathbb{E}_p \boldsymbol{X}_h = \boldsymbol{F},$$

$$\boldsymbol{X}_{1h} + g\mathbf{1} - \boldsymbol{X}_{2h} \geq 0, \quad \boldsymbol{X}_{3h} \geq 0, \quad (\boldsymbol{X}_{1h} + g\mathbf{1} - \boldsymbol{X}_{2h}) \cdot \boldsymbol{X}_{3h} = 0.$$

$$\mathbb{E}_p := \begin{bmatrix} \mu_1 \mathbb{S} & \mathbf{0} & -\mathbb{I}_{\mathrm{d}} \\ \mathbf{0} & \mu_2 \mathbb{S} & +\mathbb{I}_{\mathrm{d}} \end{bmatrix}$$

Resolution

C-functions

Definition

$f : (\mathbb{R}^m)^2 \to \mathbb{R}^m$ ($m \geq 1$) is a *C*-function or a complementarity function if

$$\forall (\boldsymbol{x}, \boldsymbol{y}) \in (\mathbb{R}^m)^2 \qquad f(\boldsymbol{x}, \boldsymbol{y}) = \boldsymbol{0} \quad \iff \quad \boldsymbol{x} \geq \boldsymbol{0}, \quad \boldsymbol{y} \geq \boldsymbol{0}, \quad \boldsymbol{x} \cdot \boldsymbol{y} = 0.$$

Examples of *C*-functions are the min function

$$(\min\{\boldsymbol{x}, \boldsymbol{y}\})_l := \min\{\boldsymbol{x}_l, \boldsymbol{y}_l\} \qquad l = 1, \ldots, m,$$

the Fischer–Burmeister function

$$(f_{\mathrm{FB}}(\boldsymbol{x}, \boldsymbol{y}))_l := \sqrt{\boldsymbol{x}_l^2 + \boldsymbol{y}_l^2} - (\boldsymbol{x}_l + \boldsymbol{y}_l) \quad l = 1, \ldots, m,$$

or the Mangasarian function

$$(f_{\mathrm{M}}(\boldsymbol{x}, \boldsymbol{y}))_l := \xi(|\boldsymbol{x}_l - \boldsymbol{y}_l|) - \xi(\boldsymbol{y}_l) - \xi(\boldsymbol{x}_l) \quad l = 1, \ldots, m,$$

where $\xi : \mathbb{R} \mapsto \mathbb{R}$ is an increasing function satisfying $\xi(\boldsymbol{0}) = \boldsymbol{0}$.

Let $\tilde{\mathbf{C}}$ be any C-function, *i.e.*, satisfying (for $m = \mathcal{N}_{\mathrm{d}}^{p,\mathrm{int}}$)

$$\tilde{\mathbf{C}}(\boldsymbol{X}_{1h} + g\mathbf{1} - \boldsymbol{X}_{2h}, \boldsymbol{X}_{3h}) = 0 \iff \boldsymbol{X}_{1h} + g\mathbf{1} - \boldsymbol{X}_{2h} \geq 0, \boldsymbol{X}_{3h} \geq 0, \ (\boldsymbol{X}_{1h} + g\mathbf{1} - \boldsymbol{X}_{2h}) \cdot X_{3h} = 0.$$

Then, introducing the function $\mathbf{C} : \mathbb{R}^{3\mathcal{N}_{\mathrm{d}}^{p,\mathrm{int}}} \to \mathbb{R}^{\mathcal{N}_{\mathrm{d}}^{p,\mathrm{int}}}$ defined as

$$\mathbf{C}(\boldsymbol{X}_h) = \tilde{\mathbf{C}}(\boldsymbol{X}_{1h} + g\mathbf{1} - \boldsymbol{X}_{2h}, \boldsymbol{X}_{3h})$$

Our problem can be equivalently rewritten as

$$\begin{cases} \mathbb{E}_p \boldsymbol{X}_h & = \boldsymbol{F}, \\ \mathbf{C}(\boldsymbol{X}_h) & = \mathbf{0}. \end{cases}$$

The C-function is not Fréchet differentiable.

How can we solve the nonlinear problem?

Inexact semismooth Newton method

Newton initial vector: $\boldsymbol{X}_h^0 := \left(\boldsymbol{X}_{1h}^0, \boldsymbol{X}_{2h}^0, \boldsymbol{X}_{3h}^0\right)^T \in \mathbb{R}^{3\mathcal{N}_{\mathrm{d}}^{p,\mathrm{int}}}$, on step $k \geq 1$, one looks for $\boldsymbol{X}_h^k \in \mathbb{R}^{3\mathcal{N}_{\mathrm{d}}^{p,\mathrm{int}}}$ such that

$$\mathbb{A}^{k-1}\boldsymbol{X}_h^k = \boldsymbol{B}^{k-1},$$

where

$$\mathbb{A}^{k-1} := \begin{bmatrix} \mathbb{E}_p \\ \mathbf{J}_{\mathbf{C}}(\boldsymbol{X}_h^{k-1}) \end{bmatrix}, \quad \boldsymbol{B}^{k-1} := \begin{bmatrix} \boldsymbol{F} \\ \mathbf{J}_{\mathbf{C}}(\boldsymbol{X}_h^{k-1})\boldsymbol{X}_h^{k-1} - \mathbf{C}(\boldsymbol{X}_h^{k-1}) \end{bmatrix}.$$

Inexact semismooth Newton method

Newton initial vector: $\boldsymbol{X}_h^0 := \left(\boldsymbol{X}_{1h}^0, \boldsymbol{X}_{2h}^0, \boldsymbol{X}_{3h}^0\right)^T \in \mathbb{R}^{3\mathcal{N}_{\mathrm{d}}^{p,\mathrm{int}}}$, on step $k \geq 1$, one looks for $\boldsymbol{X}_h^k \in \mathbb{R}^{3\mathcal{N}_{\mathrm{d}}^{p,\mathrm{int}}}$ such that

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Inexact solver initial vector: $\boldsymbol{X}_h^{k,0} \in \mathbb{R}^{3\mathcal{N}_{\mathrm{d}}^{p,\mathrm{int}}}$, often taken as $\boldsymbol{X}_h^{k,0} = \boldsymbol{X}_h^{k-1}$, this yields on step $i \geq 1$ an approximation $\boldsymbol{X}_h^{k,i}$ to $\boldsymbol{X}_h^k$ satisfying

$$\mathbb{A}^{k-1}\boldsymbol{X}_h^{k,i} = \boldsymbol{B}^{k-1} - \boldsymbol{R}_h^{k,i},$$

where $\boldsymbol{R}_h^{k,i} \in \mathbb{R}^{3\mathcal{N}_{\mathrm{d}}^{p,\mathrm{int}}}$ is the algebraic residual vector.

Inexact semismooth Newton method

A posteriori error estimates

A posteriori analysis

$$\left|\!\left|\!\left| \boldsymbol{u} - \boldsymbol{u}_h^{k,i} \right|\!\right|\!\right|_\Omega := \left(\sum_{\alpha=1}^{2} \mu_\alpha \left\| \nabla \left(u_\alpha - u_{\alpha h}^{k,i} \right) \right\|_\Omega^2 \right)^{\frac{1}{2}} \leq \eta^{k,i} := \left(\sum_{K \in Th} \left[\eta_K(\boldsymbol{u}_h^{k,i})^2 \right]^2 \right)^{\frac{1}{2}}$$

- $\eta_K(\boldsymbol{u}_h^{k,i})$ local estimator depending on the approximate solution
- $\eta^{k,i} \leq \eta_{\text{disc}}^{k,i} + \eta_{\text{lin}}^{k,i} + \eta_{\text{alg}}^{k,i}$: identification of the error components
- $\eta_K(\boldsymbol{u}_h^{k,i}) \leq \left|\!\left|\!\left| \boldsymbol{u} - \boldsymbol{u}_h^{k,i} \right|\!\right|\!\right|_{\zeta_K}$: local efficiency
- adaptive inexact stopping criteria based on the error components

A posteriori analysis

$$\left|\left|\left|\boldsymbol{u}-\boldsymbol{u}_h^{k,i}\right|\right|\right|_\Omega := \left(\sum_{\alpha=1}^{2}\mu_\alpha\left\|\nabla\left(u_\alpha-u_{\alpha h}^{k,i}\right)\right\|_\Omega^2\right)^{\frac{1}{2}} \leq \eta^{k,i} := \left(\sum_{K\in Th}\left[\eta_K(\boldsymbol{u}_h^{k,i})^2\right]^2\right)^{\frac{1}{2}}$$

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- $\eta^{k,i} \leq \eta_{\mathrm{disc}}^{k,i} + \eta_{\mathrm{lin}}^{k,i} + \eta_{\mathrm{alg}}^{k,i}$: identification of the error components
- $\eta_K(\boldsymbol{u}_h^{k,i}) \leq \left|\left|\left|\boldsymbol{u}-\boldsymbol{u}_h^{k,i}\right|\right|\right|_{\zeta_K}$: local efficiency
- adaptive inexact stopping criteria based on the error components

We employ the methodology of equilibrated flux reconstruction to obtain local error estimators.

Destuynder & Métivet (1999) Braess & Schöberl (2008), Ern & Vohralík (2013)

Component flux reconstruction

Motivation:

$$-\mu_\alpha \nabla u_\alpha \in \mathbf{H}(\mathrm{div},\Omega), \quad -\mu_\alpha \nabla u_{\alpha h}^{k,i} \notin \mathbf{H}(\mathrm{div},\Omega), \quad \nabla \cdot \left(-\mu_\alpha \nabla u_{\alpha h}^{k,i}\right) \neq f_\alpha - (-1)^\alpha \lambda_h^{k,i}$$

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Flux reconstruction:

$$\boldsymbol{\sigma}_{\alpha h}^{k,i} \in \mathbf{H}(\mathrm{div}, \Omega) \quad \left(\nabla \cdot \boldsymbol{\sigma}_{\alpha h}^{k,i}, 1\right)_K = \left(f_\alpha - (-1)^\alpha \lambda_h^{k,i}, 1\right)_K$$

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Decomposition of the flux:

$$\boldsymbol{\sigma}_{\alpha h}^{k,i} = \boldsymbol{\sigma}_{\alpha h,\mathrm{alg}}^{k,i} + \boldsymbol{\sigma}_{\alpha h,\mathrm{disc}}^{k,i}$$

Component flux reconstruction

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Decomposition of the flux:

$$\boldsymbol{\sigma}_{\alpha h}^{k,i} = \boldsymbol{\sigma}_{\alpha h,\mathrm{alg}}^{k,i} + \boldsymbol{\sigma}_{\alpha h,\mathrm{disc}}^{k,i}$$

Algebraic flux reconstruction:

$\boldsymbol{\sigma}_{\alpha h,\mathrm{alg}}^{k,i} \in \mathbf{H}(\mathrm{div},\Omega) \quad \nabla\cdot\boldsymbol{\sigma}_{\alpha h,\mathrm{alg}}^{k,i} = r_{\alpha h}^{k,i}$ where $r_{\alpha h}^{k,i}$ is the functional representation of $\boldsymbol{R}_{\alpha h}^{k,i}$

Papež, Rüde, Vohralík and Wohlmuth. Submitted for publication (2017).

Component flux reconstruction

Motivation:

$$-\mu_\alpha \nabla u_\alpha \in \mathbf{H}(\mathrm{div}, \Omega), \quad -\mu_\alpha \nabla u_{\alpha h}^{k,i} \notin \mathbf{H}(\mathrm{div}, \Omega), \quad \nabla \cdot \left(-\mu_\alpha \nabla u_{\alpha h}^{k,i}\right) \neq f_\alpha - (-1)^\alpha \lambda_h^{k,i}$$

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Discretization flux reconstruction:

$$\boldsymbol{\sigma}_{\alpha h,\mathrm{disc}}^{k,i} \in \mathbf{H}(\mathrm{div}, \Omega) \quad \left(\nabla \cdot \boldsymbol{\sigma}_{\alpha h,\mathrm{disc}}^{k,i}, 1\right)_K = \left(f_\alpha - (-1)^\alpha \lambda_h^{k,i} - r_{\alpha h}^{k,i}, 1\right)_K$$

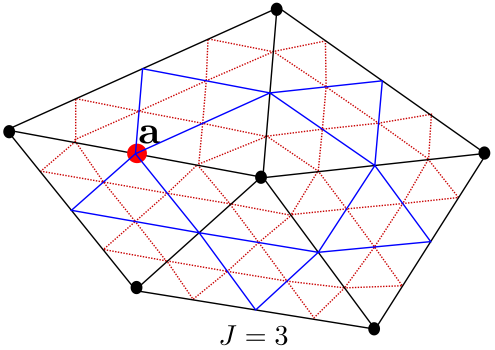
Discretization flux reconstruction

$$\left(\boldsymbol{\sigma}_{\alpha h,\mathrm{disc}}^{k,i,\boldsymbol{a}}, \boldsymbol{\tau}_h\right)_{\omega_h^{\boldsymbol{a}}} - \left(\gamma_{\alpha h}^{k,i,\boldsymbol{a}}, \nabla\cdot\boldsymbol{\tau}_h\right)_{\omega_h^{\boldsymbol{a}}} = -\left(\mu_\alpha \psi_{h,\boldsymbol{a}} \nabla u_{\alpha h}^{k,i,\boldsymbol{a}}, \boldsymbol{\tau}_h\right)_{\omega_h^{\boldsymbol{a}}} \quad \forall \boldsymbol{\tau}_h \in \mathbf{V}_h^{\boldsymbol{a}},$$
$$\left(\nabla\cdot\boldsymbol{\sigma}_{\alpha h,\mathrm{disc}}^{k,i,\boldsymbol{a}}, q_h\right)_{\omega_h^{\boldsymbol{a}}} = \left(\tilde{g}_{\alpha h}^{k,i,\boldsymbol{a}}, q_h\right)_{\omega_h^{\boldsymbol{a}}} \quad \forall q_h \in Q_h^{\boldsymbol{a}},$$

$\boldsymbol{a} \in \mathcal{V}_h^{\mathrm{int}}$

$\mathbf{V}_h^{\boldsymbol{a}} := \left\{\boldsymbol{\tau}_h \in \mathbf{RT}_p(\omega_h^{\boldsymbol{a}}),\ \boldsymbol{\tau}_h\cdot\mathbf{n}_{\omega_h^{\boldsymbol{a}}} = 0 \text{ on } \partial\omega_h^{\boldsymbol{a}}\right\}$

$Q_h^{\boldsymbol{a}} := \mathbb{P}_p^0(\omega_h^{\boldsymbol{a}})$



Discretization flux reconstruction

$$\left(\boldsymbol{\sigma}_{\alpha h,\mathrm{disc}}^{k,i,\boldsymbol{a}}, \boldsymbol{\tau}_h\right)_{\omega_h^{\boldsymbol{a}}} - \left(\gamma_{\alpha h}^{k,i,\boldsymbol{a}}, \nabla\cdot\boldsymbol{\tau}_h\right)_{\omega_h^{\boldsymbol{a}}} = -\left(\mu_\alpha \psi_{h,\boldsymbol{a}} \nabla u_{\alpha h}^{k,i,\boldsymbol{a}}, \boldsymbol{\tau}_h\right)_{\omega_h^{\boldsymbol{a}}} \quad \forall \boldsymbol{\tau}_h \in \mathbf{V}_h^{\boldsymbol{a}},$$

$$\left(\nabla\cdot\boldsymbol{\sigma}_{\alpha h,\mathrm{disc}}^{k,i,\boldsymbol{a}}, q_h\right)_{\omega_h^{\boldsymbol{a}}} = \left(\tilde{g}_{\alpha h}^{k,i,\boldsymbol{a}}, q_h\right)_{\omega_h^{\boldsymbol{a}}} \quad \forall q_h \in Q_h^{\boldsymbol{a}},$$

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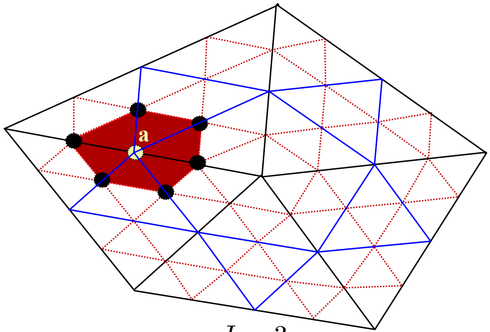
$\mathbf{V}_h^{\boldsymbol{a}} := \left\{\boldsymbol{\tau}_h \in \mathbf{RT}_p(\omega_h^{\boldsymbol{a}}),\ \boldsymbol{\tau}_h\cdot\mathbf{n}_{\omega_h^{\boldsymbol{a}}} = 0 \text{ on } \partial\omega_h^{\boldsymbol{a}}\right\}$

$Q_h^{\boldsymbol{a}} := \mathbb{P}_p^0(\omega_h^{\boldsymbol{a}})$

$$\boldsymbol{\sigma}_{\alpha h,\mathrm{disc}}^{k,i} := \sum_{\boldsymbol{a}\in\mathcal{V}_h} \boldsymbol{\sigma}_{\alpha h,\mathrm{disc}}^{k,i,\boldsymbol{a}}$$



$J = 3$

Estimators

Violations of physical properties of the numerical solution

$$\boldsymbol{\sigma}_{\alpha h}^{k,i} \neq -\nabla u_{\alpha h}^{k,i} \quad \nabla \cdot \left(-\mu_\alpha \nabla u_{\alpha h}^{k,i}\right) \neq f_\alpha - (-1)^\alpha \lambda_h^{k,i}$$

Estimators

Violations of physical properties of the numerical solution

$$\boldsymbol{\sigma}_{\alpha h}^{k,i} \neq -\nabla u_{\alpha h}^{k,i} \quad \nabla \cdot \left(-\mu_\alpha \nabla u_{\alpha h}^{k,i}\right) \neq f_\alpha - (-1)^\alpha \lambda_h^{k,i}$$

Flux estimator:

$$\eta_{\mathrm{F},K,\alpha}^{k,i} := \left\| \mu_\alpha^{\frac{1}{2}} \nabla u_{\alpha h}^{k,i} + \mu_\alpha^{-\frac{1}{2}} \boldsymbol{\sigma}_{\alpha h}^{k,i} \right\|_K,$$

Residual estimator:

$$\eta_{\mathrm{R},K,\alpha}^{k,i} := \frac{h_K}{\pi} \mu_\alpha^{-\frac{1}{2}} \left\| f_\alpha - \nabla \cdot \boldsymbol{\sigma}_{\alpha h}^{k,i} - (-1)^\alpha \lambda_h^{k,i} \right\|_K,$$

Estimators

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Violations of the complementarity constraints

$$p = 1 : (u_{1h}^{k,i} - u_{2h}^{k,i})(\boldsymbol{a}) \ngeq 0 \quad \lambda_h^{k,i}(\boldsymbol{a}) \ngeq 0 \quad \lambda_h^{k,i}(\boldsymbol{a}) \cdot (u_{1h}^{k,i} - u_{2h}^{k,i})(\boldsymbol{a}) \neq 0 \quad \forall \boldsymbol{a} \in \mathcal{V}_h^{\mathrm{int}}$$

$$p \geq 2 : (u_{1h}^{k,i} - u_{2h}^{k,i})(\boldsymbol{x}_l) \ngeq 0 \,, \; \left(\lambda_h^{k,i}, \psi_{h,\boldsymbol{x}_l}\right)_\Omega \ngeq 0 \; \forall \boldsymbol{x}_l \in \mathcal{V}_{\mathrm{d}}^{p,\mathrm{int}} \; \left(\lambda_h^{k,i}, u_{1h}^{k,i} - u_{2h}^{k,i}\right)_\Omega \neq 0$$

Strategy for constructing the estimators

$$\lambda_h^{k,i} := \lambda_h^{k,i,\mathrm{pos}} + \lambda_h^{k,i,\mathrm{neg}}, \quad \widetilde{\mathcal{K}}_{gh}^p := \left\{(v_{1h}, v_{2h}) \in X_{gh}^p \times X_{0h}^p,\ v_{1h} - v_{2h} \geq 0\right\} \subset \mathcal{K}_g.$$

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Contact estimator:

$$\eta_{\mathrm{C},K}^{k,i,\mathrm{pos}} := 2\left(\lambda_h^{k,i,\mathrm{pos}}, u_{1h}^{k,i} - u_{2h}^{k,i}\right)_K,$$

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Nonconformity estimator 1:

$$\eta_{\mathrm{nonc},1,K}^{k,i} := \left|\left|\left| \boldsymbol{s}_h^{k,i} - \boldsymbol{u}_h^{k,i} \right|\right|\right|_K,$$

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Nonconformity estimator 2:

$$\eta_{\mathrm{nonc},2,K}^{k,i} := h_\Omega C_{\mathrm{PF}} \left(\frac{1}{\mu_1} + \frac{1}{\mu_2} \right)^{\frac{1}{2}} \left\| \lambda_h^{k,i,\mathrm{neg}} \right\|_K,$$

Strategy for constructing the estimators

$$\lambda_h^{k,i} := \lambda_h^{k,i,\mathrm{pos}} + \lambda_h^{k,i,\mathrm{neg}}, \quad \widetilde{\mathcal{K}}_{gh}^{p} := \left\{ (v_{1h}, v_{2h}) \in X_{gh}^{p} \times X_{0h}^{p},\ v_{1h} - v_{2h} \geq 0 \right\} \subset \mathcal{K}_g.$$

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Nonconformity estimator 1:

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Nonconformity estimator 2:

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Nonconformity estimator 3:

$$\eta_{\mathrm{nonc},3,K}^{k,i} := 2 h_\Omega C_{\mathrm{PF}} \left(\frac{1}{\mu_1} + \frac{1}{\mu_2} \right)^{\frac{1}{2}} \left\| \lambda_h^{k,i,\mathrm{pos}} \right\|_\Omega \left|\left|\left| \boldsymbol{s}_h^{k,i} - \boldsymbol{u}_h^{k,i} \right|\right|\right|_K.$$

Theorem (A posteriori error estimate)

$$\left|\!\left|\!\left| \boldsymbol{u} - \boldsymbol{u}_h^{k,i} \right|\!\right|\!\right| \leq \left\{ \left(\left(\sum_{K\in\mathcal{T}_h} \sum_{\alpha=1}^{2} \left(\eta_{\mathrm{F},K,\alpha}^{k,i} + \eta_{\mathrm{R},K,\alpha}^{k,i} \right)^2 \right)^{\frac{1}{2}} + \eta_{\mathrm{nonc},1}^{k,i} + \eta_{\mathrm{nonc},2}^{k,i} \right)^2 + \eta_{\mathrm{nonc},3}^{k,i} + \sum_{K\in\mathcal{T}_h} \eta_{\mathrm{C},K}^{k,i,\mathrm{pos}} \right\}^{\frac{1}{2}}$$

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Corollary (Distinction of the error components)

$$\left\|\left| \boldsymbol{u} - \boldsymbol{u}_h^{k,i} \right|\right\| \leq \eta_{\mathrm{disc}}^{k,i} + \eta_{\mathrm{lin}}^{k,i} + \eta_{\mathrm{alg}}^{k,i}$$

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Adaptive algorithm

If $\eta_{\mathrm{alg}}^{k,i} \leq \gamma_{\mathrm{alg}} \max \left\{ \eta_{\mathrm{disc}}^{k,i}, \eta_{\mathrm{lin}}^{k,i} \right\}$

Stop linear solver

If $\eta_{\mathrm{lin}}^{k,i} \leq \gamma_{\mathrm{lin}} \eta_{\mathrm{disc}}^{n,k,i}$

Stop nonlinear solver

Theorem (A posteriori error estimate)

$$\left|\!\left|\!\left| \boldsymbol{u} - \boldsymbol{u}_h^{k,i} \right|\!\right|\!\right| \leq \left\{ \left(\left(\sum_{K\in\mathcal{T}_h} \sum_{\alpha=1}^{2} \left(\eta_{\mathrm{F},K,\alpha}^{k,i} + \eta_{\mathrm{R},K,\alpha}^{k,i} \right)^2 \right)^{\frac{1}{2}} + \eta_{\mathrm{nonc},1}^{k,i} + \eta_{\mathrm{nonc},2}^{k,i} \right)^2 + \eta_{\mathrm{nonc},3}^{k,i} + \sum_{K\in\mathcal{T}_h} \eta_{\mathrm{C},K}^{k,i,\mathrm{pos}} \right\}^{\frac{1}{2}}$$

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Adaptive algorithm

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Stop linear solver

If $\eta_{\mathrm{lin}}^{k,i} \leq \gamma_{\mathrm{lin}} \eta_{\mathrm{disc}}^{n,k,i}$

Stop nonlinear solver

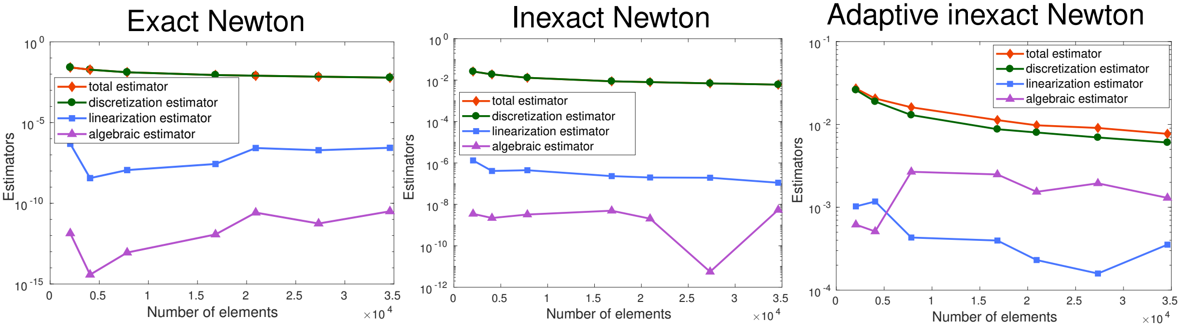
Local efficiency (shown only for p=1):

$$\eta_{\mathrm{disc},K}^{k,i} \lesssim \sum_{\boldsymbol{a}\in\mathcal{V}_h} \left(\left\| \mu_\alpha^{\frac{1}{2}} \nabla \left(u_\alpha - u_{\alpha h}^{k,i} \right) \right\|_{\omega_h^{\boldsymbol{a}}} + \left|\!\left|\!\left| \lambda - \lambda_h^{k,i}(\boldsymbol{a}) \right|\!\right|\!\right|_{H_*^{-1}(\omega_h^{\boldsymbol{a}})} \right)$$

Numerical experiments

Numerical experiments

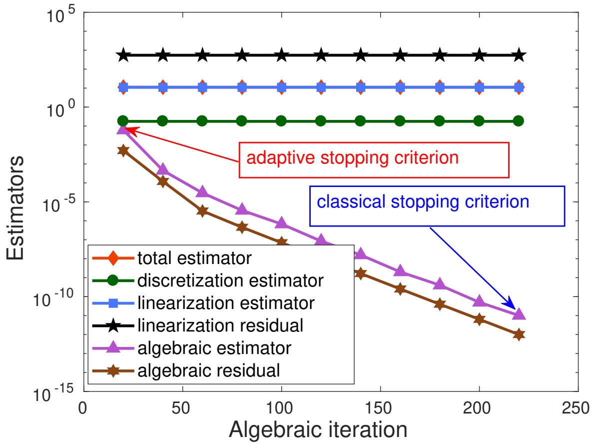
- Ω = unit disk, $J = 3$, $\mu_1 = \mu_2 = 1$, $g = 0.05$, $\gamma_{\text{lin}} = 0.3$ $\gamma_{\text{alg}} = 0.3$
- semismooth solver: Newton-min. Linear solver: GMRES with ILU preconditionner.



Quality and precision are preserved for adaptive inexact semismooth Newton method.

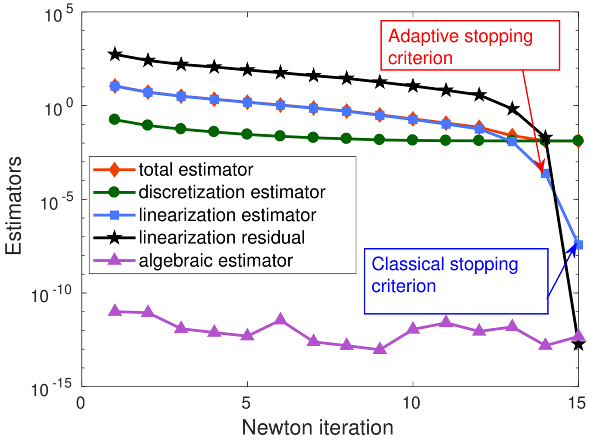
GMRES adaptivity

Exact/Adapt inexact Newton

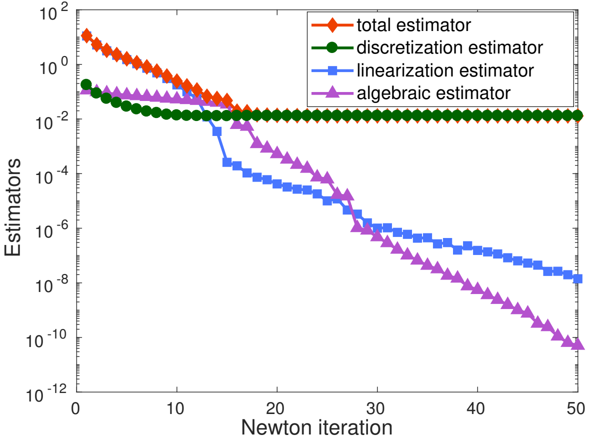


Newton-min adaptivity

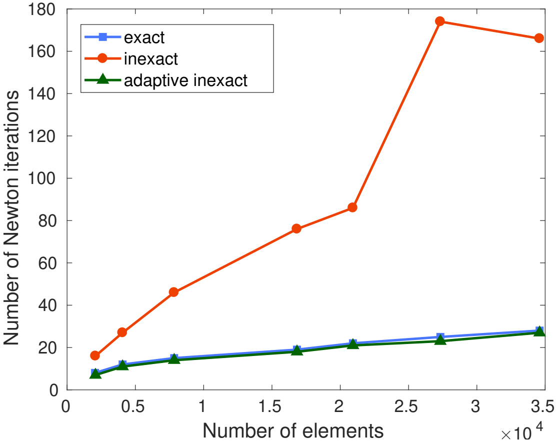
Exact/Adapt inexact Newton

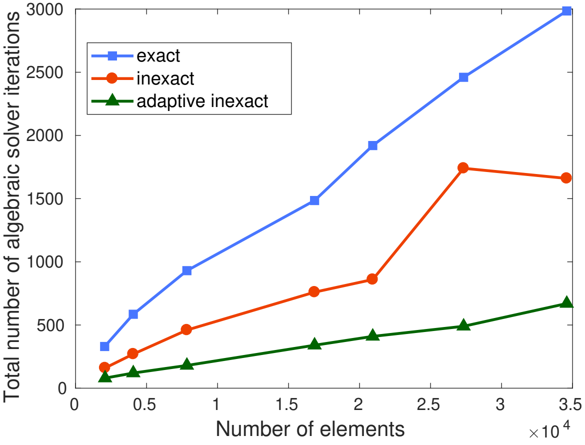


Inexact Newton

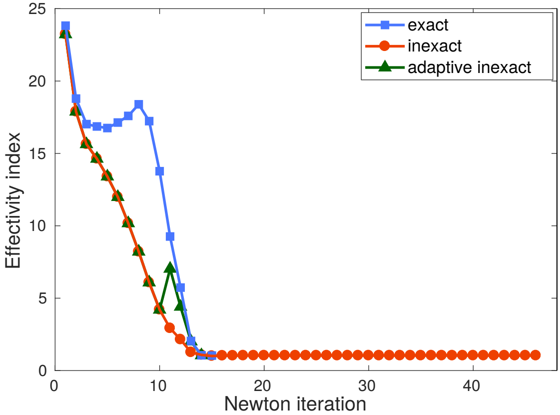


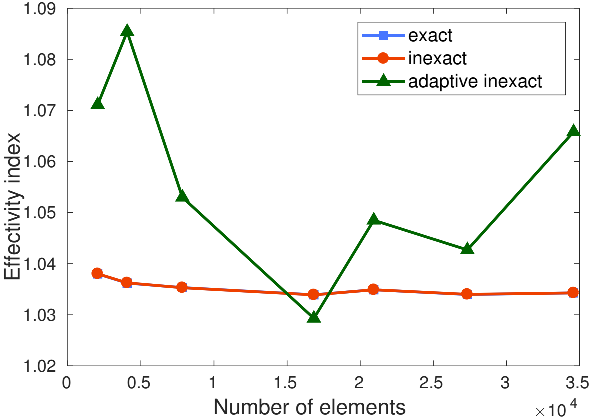
Overall performance



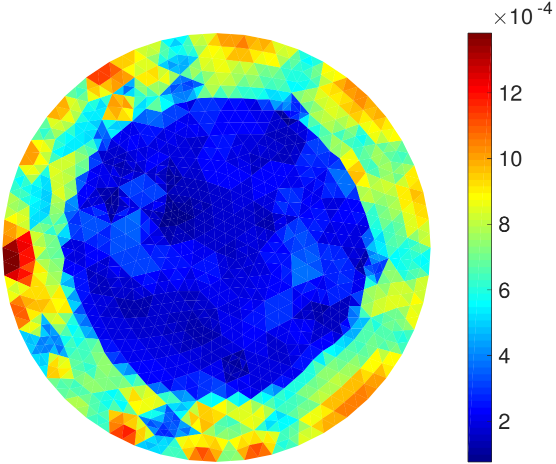


Effectivity indices

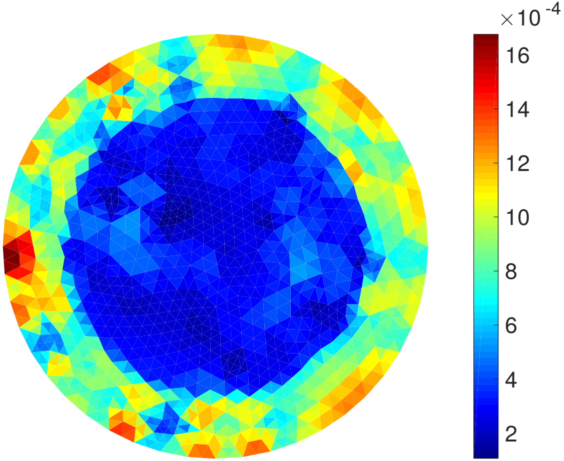




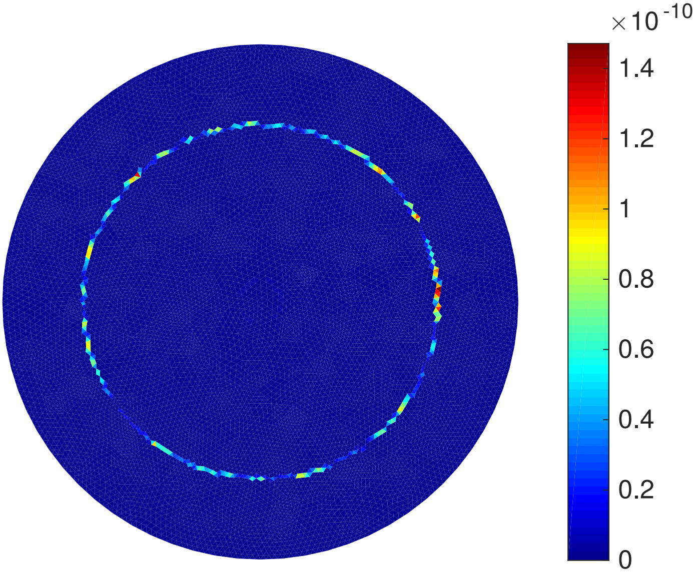
Local distribution of the error



Local error estimator



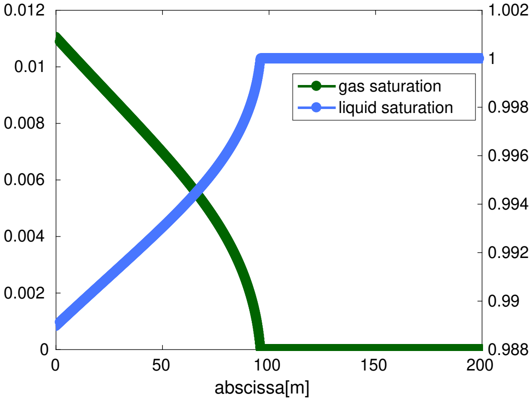
Contact estimator

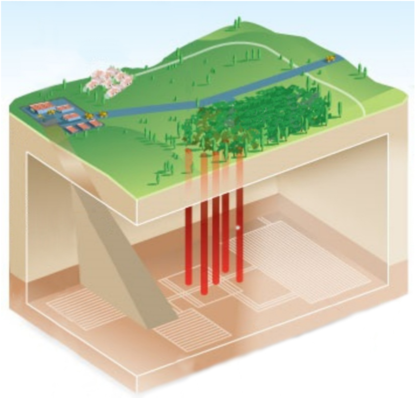


Outline

Model problem for the storage of radioactive wastes

$$\begin{cases} \partial_t l_{\rm w}(S^{\rm l}) + \nabla \cdot \Phi_{\rm w}(S^{\rm l}, P^{\rm l}, \chi_{\rm h}^{\rm l}) = Q_{\rm w}, \\ \partial_t l_{\rm h}(S^{\rm l}, P^{\rm l}, \chi_{\rm h}^{\rm l}) + \nabla \cdot \Phi_{\rm h}(S^{\rm l}, P^{\rm l}, \chi_{\rm h}^{\rm l}) = Q_{\rm h}, \\ 1 - S^{\rm l} \geq 0, \ H\left(P^{\rm l} + P_{\rm cp}(S^{\rm l})\right) - \beta_{\rm l}\chi_{\rm h}^{\rm l} \geq 0, \ (1 - S^{\rm l}) \cdot \left(H\left(P^{\rm l} + P_{\rm cp}(S^{\rm l})\right) - \beta_{\rm l}\chi_{\rm h}^{\rm l}\right) = 0 \end{cases}$$





Discretization by the finite volume method

Numerical solution:

$$\boldsymbol{U}^n := (\boldsymbol{U}_K^n)_{K\in\mathcal{T}_h}, \qquad \boldsymbol{U}_K^n := (S_K^n, P_K^n, \chi_K^n)$$ **one value per cell and time step**

Discretization of the water equation

$$S_{\mathrm{w},K}^n(\boldsymbol{U}^n) := |K|\partial_t^n l_{\mathrm{w},K} + \sum_{\sigma\in\mathcal{E}_K} F_{\mathrm{w},K,\sigma}(\boldsymbol{U}^n) - |K|Q_{\mathrm{w,K}}^n = 0,$$

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Discretization by the finite volume method

Numerical solution:

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Discretization of the hydrogen equation

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At each time step, for each components, we obtain the nonlinear system of algebraic equations

$$S_{c,K}^n(\boldsymbol{U}_h^n) = 0$$

Discrete complementarity problem and semismoothness

Discretization of the nonlinear complementarity constraints

$$\mathcal{K}(\boldsymbol{U}_K^n) := 1 - S_K^n \quad \mathcal{G}(\boldsymbol{U}_K^n) := H(P_K^n + P_{\mathrm{cp}}(S_K^n)) - \beta^{\mathrm{l}} \chi_K^n$$

The discretization reads

$$S_{c,K}^n(\boldsymbol{U}_h^n) = 0$$

$$\mathcal{K}(\boldsymbol{U}_K^n) \geq 0, \quad \mathcal{G}(\boldsymbol{U}_K^n) \geq 0, \quad \mathcal{K}(\boldsymbol{U}_K^n) \cdot \mathcal{G}(\boldsymbol{U}_K^n) = 0$$

- **We reformulate the complementarity constraints with C-functions**
- **We employ inexact semismooth linearization**
- **Can we estimate each error component?**

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A posteriori error estimates

Weak solution

Motivation: Define rigorously the dual norm of the residual

$$\left\|\mathcal{R}_c(S^{n,k,i}_{h\tau}, P^{n,k,i}_{h\tau}, \chi^{n,k,i}_{h\tau})\right\|_{X'_n} := \sup_{\substack{\varphi\in X_n\\ \|\varphi\|_{X_n}=1}} \int_{I_n} \left(Q_c - \partial_t l^{n,k,i}_{c,h\tau}, \varphi\right)_\Omega (t) + \left(\mathbf{\Phi}^{n,k,i}_{c,h\tau}, \nabla\varphi\right)_\Omega (t)\,\mathrm{dt}$$

$$X := L^2((0,t_{\mathrm{F}}); H^1(\Omega)),\ \ Y := H^1((0,t_{\mathrm{F}}); L^2(\Omega)),\ \ Z := L^2_+((0,t_{\mathrm{F}}); L^\infty(\Omega))$$

$$\|\varphi\|_{X_n} := \int_{I_n} \sum_{K\in\mathcal{T}_h} \|\varphi\|^2_{X,K}\,\mathrm{dt},\ \ \|\varphi\|^2_{X,K} := \varepsilon h_K^{-2} \|\varphi\|^2_K + \|\nabla\varphi\|^2_K$$

Assumption:

- $1 - S^{\mathrm{l}} \in Z,\ \ l_c \in Y,\ \ P^{\mathrm{l}} \in X,\ \ \chi^{\mathrm{l}}_{\mathrm{h}} \in X,\ \ \mathbf{\Phi}_c \in L^2((0,t_{\mathrm{F}}); \mathbf{H}(\mathrm{div},\Omega))$
- $\int_0^{t_{\mathrm{F}}} (\partial_t l_c, \varphi)_\Omega (t)\,\mathrm{dt} - \int_0^{t_{\mathrm{F}}} (\mathbf{\Phi}_c, \nabla\varphi)_\Omega (t)\,\mathrm{dt} = \int_0^{t_{\mathrm{F}}} (Q_c, \varphi)_\Omega (t)\,\mathrm{dt} \quad \forall\varphi\in X$
- $\int_0^{t_{\mathrm{F}}} \left(\lambda - (1 - S^{\mathrm{l}}), H[P^{\mathrm{l}} + P_{\mathrm{cp}}(S^{\mathrm{l}})] - \beta^{\mathrm{l}}\chi^{\mathrm{l}}_{\mathrm{h}}\right)_\Omega (t)\,\mathrm{dt} \geq 0 \quad \forall\lambda\in Z$
- the initial condition holds

Weak solution

Motivation: Define rigorously the dual norm of the residual

$$\left\|\mathcal{R}_c(S_{h\tau}^{n,k,i}, P_{h\tau}^{n,k,i}, \chi_{h\tau}^{n,k,i})\right\|_{X_n'} := \sup_{\substack{\varphi\in X_n \\ \|\varphi\|_{X_n}=1}} \int_{I_n} \left(Q_c - \partial_t l_{c,h\tau}^{n,k,i}, \varphi\right)_\Omega (t) + \left(\boldsymbol{\Phi}_{c,h\tau}^{n,k,i}, \nabla\varphi\right)_\Omega (t)\,\mathrm{d}t$$

$$X := L^2((0,t_\mathrm{F}); H^1(\Omega)),\ \ Y := H^1((0,t_\mathrm{F}); L^2(\Omega)),\ \ Z := L^2_+((0,t_\mathrm{F}); L^\infty(\Omega))$$

$$\|\varphi\|_{X_n} := \int_{I_n} \sum_{K\in\mathcal{T}_h} \|\varphi\|_{X,K}^2\,\mathrm{d}t,\ \ \|\varphi\|_{X,K}^2 := \varepsilon h_K^{-2}\|\varphi\|_K^2 + \|\nabla\varphi\|_K^2$$

Assumption:

- $1 - S^\mathrm{l} \in Z,\ \ l_c \in Y,\ \ P^\mathrm{l} \in X,\ \ \chi_\mathrm{h}^\mathrm{l} \in X,\ \ \boldsymbol{\Phi}_c \in L^2((0,t_\mathrm{F}); \mathbf{H}(\mathrm{div},\Omega))$
- $\int_0^{t_\mathrm{F}} (\partial_t l_c, \varphi)_\Omega (t)\,\mathrm{d}t - \int_0^{t_\mathrm{F}} (\boldsymbol{\Phi}_c, \nabla\varphi)_\Omega (t)\,\mathrm{d}t = \int_0^{t_\mathrm{F}} (Q_c, \varphi)_\Omega (t)\,\mathrm{d}t \quad \forall\varphi \in X$
- $\int_0^{t_\mathrm{F}} \left(\lambda - \left(1 - S^\mathrm{l}\right), H[P^\mathrm{l} + P_\mathrm{cp}(S^\mathrm{l})] - \beta^\mathrm{l}\chi_\mathrm{h}^\mathrm{l}\right)_\Omega (t)\,\mathrm{d}t \geq 0 \quad \forall\lambda \in Z$
- the initial condition holds

Weak solution

Motivation: Define rigorously the dual norm of the residual

$$\left\|\mathcal{R}_c(S_{h\tau}^{n,k,i}, P_{h\tau}^{n,k,i}, \chi_{h\tau}^{n,k,i})\right\|_{X_n'} := \sup_{\substack{\varphi\in X_n\\ \|\varphi\|_{X_n}=1}} \int_{I_n} \left(Q_c - \partial_t l_{c,h\tau}^{n,k,i}, \varphi\right)_\Omega(t) + \left(\mathbf{\Phi}_{c,h\tau}^{n,k,i}, \nabla\varphi\right)_\Omega(t)\,\mathrm{dt}$$

$$X := L^2((0,t_\mathrm{F}); H^1(\Omega)),\ \ Y := H^1((0,t_\mathrm{F}); L^2(\Omega)),\ \ Z := L^2_+((0,t_\mathrm{F}); L^\infty(\Omega))$$

$$\|\varphi\|_{X_n} := \int_{I_n} \sum_{K\in\mathcal{T}_h} \|\varphi\|_{X,K}^2\,\mathrm{dt},\ \ \|\varphi\|_{X,K}^2 := \varepsilon h_K^{-2}\|\varphi\|_K^2 + \|\nabla\varphi\|_K^2$$

Assumption:

- $1 - S^\mathrm{l} \in Z,\ \ l_c \in Y,\ \ P^\mathrm{l} \in X,\ \ \chi_\mathrm{h}^\mathrm{l} \in X,\ \ \mathbf{\Phi}_c \in L^2((0,t_\mathrm{F}); \mathbf{H}(\mathrm{div},\Omega))$
- $\int_0^{t_\mathrm{F}} (\partial_t l_c, \varphi)_\Omega(t)\,\mathrm{dt} - \int_0^{t_\mathrm{F}} (\mathbf{\Phi}_c, \nabla\varphi)_\Omega(t)\,\mathrm{dt} = \int_0^{t_\mathrm{F}} (Q_c, \varphi)_\Omega(t)\,\mathrm{dt} \quad \forall\varphi \in X$
- $\int_0^{t_\mathrm{F}} \left(\lambda - (1 - S^\mathrm{l}), H[P^\mathrm{l} + P_\mathrm{cp}(S^\mathrm{l})] - \beta^\mathrm{l}\chi_\mathrm{h}^\mathrm{l}\right)_\Omega(t)\,\mathrm{dt} \geq 0 \quad \forall\lambda \in Z$
- the initial condition holds

Approximate solution

$$S_K^{n,k,i} \in \mathbb{P}_0^{\mathrm{d}}(\mathcal{T}_h) \quad P_K^{n,k,i} \in \mathbb{P}_0^{\mathrm{d}}(\mathcal{T}_h) \quad \chi_K^{n,k,i} \in \mathbb{P}_0^{\mathrm{d}}(\mathcal{T}_h)$$

The discrete liquid pressure and discrete molar fraction do not belong to $H^1(\Omega)$

Approximate solution

$$S_K^{n,k,i} \in \mathbb{P}_0^{\mathrm{d}}(\mathcal{T}_h) \quad P_K^{n,k,i} \in \mathbb{P}_0^{\mathrm{d}}(\mathcal{T}_h) \quad \chi_K^{n,k,i} \in \mathbb{P}_0^{\mathrm{d}}(\mathcal{T}_h)$$

The discrete liquid pressure and discrete molar fraction do not belong to $H^1(\Omega)$ **We construct a conforming solution:**

$$\mathbb{P}_0^{\mathrm{d}}(\mathcal{T}_h) \longrightarrow \mathbb{P}_2^{\mathrm{d}}(\mathcal{T}_h) \longrightarrow \mathbb{P}_2^{\mathrm{c}}(\mathcal{T}_h)$$

solving local problems in each cell | **Oswald interpolation operator**

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solving local problems in each cell | **Oswald interpolation operator**

Space-time functions:

$$S_{h\tau}^{n,k,i} \in Y, \quad P_{h\tau}^{n,k,i} \in \mathbb{P}_2^{\mathrm{d}}(\mathcal{T}_h) \notin X, \quad \chi_{h\tau}^{n,k,i} \in \mathbb{P}_2^{\mathrm{d}}(\mathcal{T}_h) \notin X$$

$$\tilde{P}_{h\tau}^{n,k,i} \in \mathbb{P}_2^{\mathrm{c}}(\mathcal{T}_h) \in X,$$

$$\tilde{\chi}_{h\tau}^{n,k,i} \in \mathbb{P}_2^{\mathrm{c}}(\mathcal{T}_h) \in X.$$

Error measure

- **Dual norm of the residual for the components**
- **Residual for the constraints**

$$\mathcal{R}_{\rm e}(S_{h\tau}^{n,k,i}, P_{h\tau}^{n,k,i}, \chi_{h\tau}^{n,k,i}) := \int_{I_n} \left(1 - S_{h\tau}^{n,k,i}, H\left[P_{h\tau}^{n,k,i} + P_{\rm cp}(S_{h\tau}^{n,k,i})\right] - \beta^{\rm l}\chi_{h\tau}^{n,k,i}\right)_{\Omega}(t)\,{\rm dt}$$

- **Error measure for the nonconformity of the pressure** $\mathcal{N}_P(P_{h\tau}^{n,k,i})$
- **Error measure for nonconformity of the molar fraction** $\mathcal{N}_\chi(\chi_{h\tau}^{n,k,i})$

$$\mathcal{N}^{n,k,i} := \left\{\sum_{c\in\mathcal{C}} \left\|\mathcal{R}_c(S_{h\tau}^{n,k,i}, P_{h\tau}^{n,k,i}, \chi_{h\tau}^{n,k,i})\right\|_{X_n'}^2\right\}^{\frac{1}{2}} + \left\{\sum_{p\in\mathcal{P}} \mathcal{N}_p^2 + \mathcal{N}_\chi^2\right\}^{\frac{1}{2}} + \mathcal{R}_{\rm e}(S_{h\tau}^{n,k,i}, P_{h\tau}^{n,k,i}, \chi_{h\tau}^{n,k,i})$$

Theorem

$$\mathcal{N}^{n,k,i} \leq \eta_{\rm disc}^{n,k,i} + \eta_{\rm lin}^{n,k,i} + \eta_{\rm alg}^{n,k,i}$$

How do we construct each error estimators?

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How do we construct each error estimators?

Component flux reconstructions

The finite volume scheme provides

$$|K|\partial_t^n l_{c,K} + \sum_{\sigma\in\mathcal{E}_K} F_{c,K,\sigma}(\boldsymbol{U}^n) = |K| Q_{c,K}^n$$

Component flux reconstructions

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$$|K|\partial_t^n l_{c,K} + \sum_{\sigma\in\mathcal{E}_K} F_{c,K,\sigma}(\boldsymbol{U}^n) = |K|Q_{c,K}^n$$

Inexact semismooth linearization

$$\frac{|K|}{\Delta t}\left[l_{c,K}\left(\boldsymbol{U}^{n,k-1}\right) - l_{c,K}^{n-1} + \mathcal{L}_{c,K}^{n,k,i}\right] + \sum_{\sigma\in\mathcal{E}_K^{\mathrm{int}}} \mathcal{F}_{c,K,\sigma}^{n,k,i} - |K|Q_{c,K}^n + \boldsymbol{R}_{c,K}^{n,k,i} = 0$$

Component flux reconstructions

The finite volume scheme provides

$$|K|\partial_t^n l_{c,K} + \sum_{\sigma\in\mathcal{E}_K} F_{c,K,\sigma}(\boldsymbol{U}^n) = |K|Q^n_{c,K}$$

Inexact semismooth linearization

$$\frac{|K|}{\Delta t}\left[l_{c,K}\left(\boldsymbol{U}^{n,k-1}\right) - l^{n-1}_{c,K} + \mathcal{L}^{n,k,i}_{c,K}\right] + \sum_{\sigma\in\mathcal{E}_K^{\mathrm{int}}} \mathcal{F}^{n,k,i}_{c,K,\sigma} - |K|Q^n_{c,K} + \boldsymbol{R}^{n,k,i}_{c,K} = 0$$

Linear perturbation in the accumulation

$$\mathcal{L}^{n,k,i}_{c,K} := \sum_{K'\in\mathcal{T}_h} \frac{|K|}{\Delta t}\frac{\partial l^n_{c,K}}{\partial \boldsymbol{U}^n_{K'}}(\boldsymbol{U}^{n,k-1}_{K'})\left[\boldsymbol{U}^{n,k,i}_{K'} - \boldsymbol{U}^{n,k-1}_{K'}\right]$$

Linearized component flux

$$\mathcal{F}^{n,k,i}_{c,K,\sigma} := \sum_{K'\in\mathcal{T}_h} \frac{\partial F_{c,K,\sigma}}{\partial \boldsymbol{U}^n_{K'}}\left(\boldsymbol{U}^{n,k-1}\right)\left[\boldsymbol{U}^{n,k,i}_{K'} - \boldsymbol{U}^{n,k-1}_{K'}\right] + F_{c,K,\sigma}\left(\boldsymbol{U}^{n,k-1}\right)$$

Discretization error flux reconstruction:

$$\Theta^{n,k,i}_{c,h,\mathrm{disc}}|_K \in \mathbf{RT}_0(K) \quad \left(\Theta^{n,k,i}_{c,h,\mathrm{disc}} \cdot \boldsymbol{n}_K, 1\right)_\sigma := F_{c,K,\sigma}\left(\boldsymbol{U}^{n,k,i}\right) \quad \forall K \in \mathcal{T}_h$$

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Linearization error flux reconstruction:

$$\boldsymbol{\Theta}^{n,k,i}_{c,h,\mathrm{lin}}|_K \in \mathbf{RT}_0(K) \quad \left(\boldsymbol{\Theta}^{n,k,i}_{c,h,\mathrm{lin}} \cdot \boldsymbol{n}_K, 1\right)_\sigma := \mathcal{F}^{n,k,i}_{c,K,\sigma} - F_{c,K,\sigma}\left(\boldsymbol{U}^{n,k,i}\right) \quad \forall K \in \mathcal{T}_h$$

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Linearization error flux reconstruction:

$$\Theta_{c,h,\mathrm{lin}}^{n,k,i}|_K \in \mathbf{RT}_0(K) \quad \left(\Theta_{c,h,\mathrm{lin}}^{n,k,i} \cdot \boldsymbol{n}_K, 1\right)_\sigma := \mathcal{F}_{c,K,\sigma}^{n,k,i} - F_{c,K,\sigma}\left(\boldsymbol{U}^{n,k,i}\right) \quad \forall K \in \mathcal{T}_h$$

Algebraic error flux reconstruction:

$$\Theta_{c,h,\mathrm{alg}}^{n,k,i,\nu} := \Theta_{c,h,\mathrm{disc}}^{n,k,i+\nu} + \Theta_{c,h,\mathrm{lin}}^{n,k,i+\nu} - \left(\Theta_{c,h,\mathrm{disc}}^{n,k,i} + \Theta_{c,h,\mathrm{lin}}^{n,k,i}\right) \quad \forall K \in \mathcal{T}_h$$

Discretization error flux reconstruction:

$$\mathbf{\Theta}^{n,k,i}_{c,h,\mathrm{disc}}|_K \in \mathbf{RT}_0(K) \quad \left(\mathbf{\Theta}^{n,k,i}_{c,h,\mathrm{disc}} \cdot \mathbf{n}_K, 1\right)_\sigma := F_{c,K,\sigma}\left(\mathbf{U}^{n,k,i}\right) \quad \forall K \in \mathcal{T}_h$$

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$$\mathbf{\Theta}^{n,k,i}_{c,h,\mathrm{lin}}|_K \in \mathbf{RT}_0(K) \quad \left(\mathbf{\Theta}^{n,k,i}_{c,h,\mathrm{lin}} \cdot \mathbf{n}_K, 1\right)_\sigma := \mathcal{F}^{n,k,i}_{c,K,\sigma} - F_{c,K,\sigma}\left(\mathbf{U}^{n,k,i}\right) \quad \forall K \in \mathcal{T}_h$$

Algebraic error flux reconstruction:

$$\mathbf{\Theta}^{n,k,i,\nu}_{c,h,\mathrm{alg}} := \mathbf{\Theta}^{n,k,i+\nu}_{c,h,\mathrm{disc}} + \mathbf{\Theta}^{n,k,i+\nu}_{c,h,\mathrm{lin}} - \left(\mathbf{\Theta}^{n,k,i}_{c,h,\mathrm{disc}} + \mathbf{\Theta}^{n,k,i}_{c,h,\mathrm{lin}}\right) \quad \forall K \in \mathcal{T}_h$$

Total flux reconstruction:

$$\mathbf{\Theta}^{n,k,i,\nu}_{c,h} := \mathbf{\Theta}^{n,k,i}_{c,h,\mathrm{disc}} + \mathbf{\Theta}^{n,k,i}_{c,h,\mathrm{lin}} + \mathbf{\Theta}^{n,k,i,\nu}_{c,h,\mathrm{alg}} \in \mathbf{H}(\mathrm{div},\Omega)$$

Error estimators

Violation of physical properties of the approximate solution

$$\partial_t l_c + \nabla \cdot \Theta_{c,h}^{n,k,i,\nu} \neq Q_c \quad \Theta_{c,h}^{n,k,i,\nu} \neq \Phi_{c,h\tau}^{n,k,i}(t^n)$$

Violation of the complementarity constraints

$$1 - S_{h\tau}^{n,k,i} \ngeq 0 \quad H\left[P_{h\tau}^{n,k,i} + P_{\mathrm{cp}}\left(S_{h\tau}^{n,k,i}\right)\right] - \beta^{\mathrm{l}}\chi_{h\tau}^{n,k,i} \ngeq 0$$

Nonconformity of the approximate solution

$$P_{h\tau}^{n,k,i} \notin X \quad \chi_{h\tau}^{n,k,i} \notin X$$

1. Discretization estimator
2. Linearization estimator
3. Algebraic estimator

Error estimators

Violation of physical properties of the approximate solution

$$\partial_t l_c + \nabla \cdot \Theta_{c,h}^{n,k,i,\nu} \neq Q_c \quad \Theta_{c,h}^{n,k,i,\nu} \neq \Phi_{c,h\tau}^{n,k,i}(t^n)$$

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Nonconformity of the approximate solution

$$P_{h\tau}^{n,k,i} \notin X \quad \chi_{h\tau}^{n,k,i} \notin X$$

1. **Discretization estimator**
2. **Linearization estimator**
3. **Algebraic estimator**

Numerical experiments

Numerical experiments

Ω: one-dimensional core with length $L = 200m$.

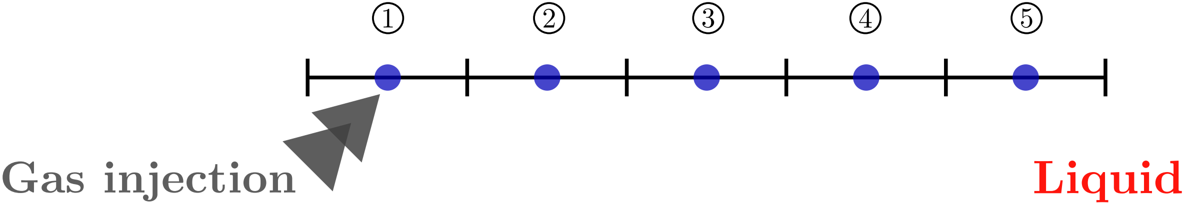
Semismooth solver: Newton-min

Iterative algebraic solver: GMRES.

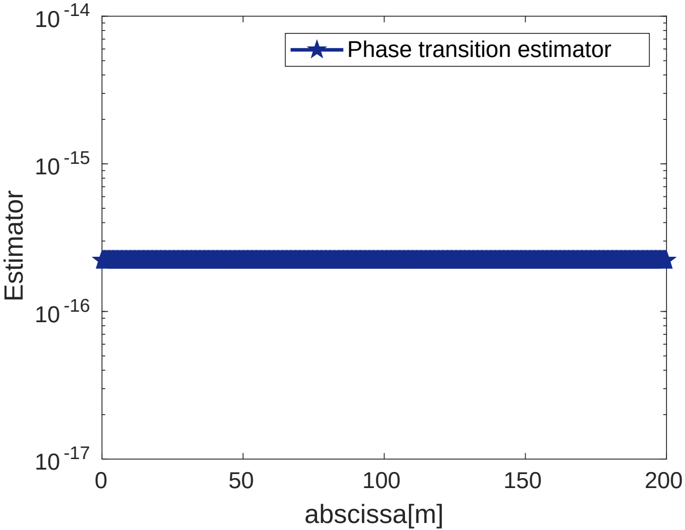
Time step: $\Delta t = 5000$ years,

Number of cells: $N_{\mathrm{sp}} = 1000$,

Final simulation time: $t_{\mathrm{F}} = 5 \times 10^5$ years.

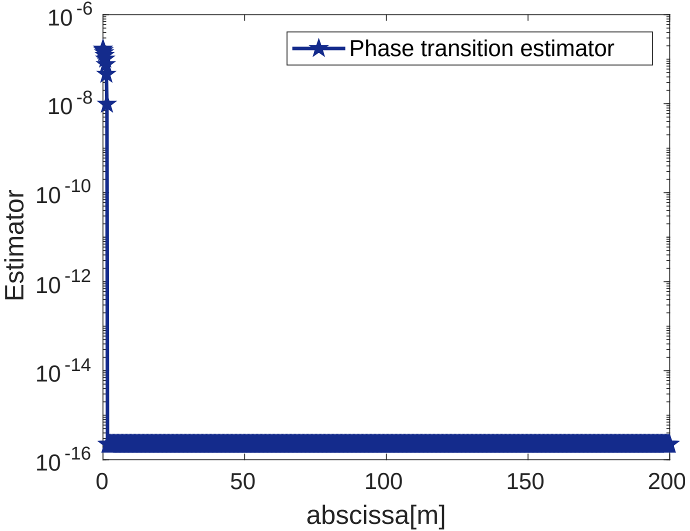


Phase transition estimator



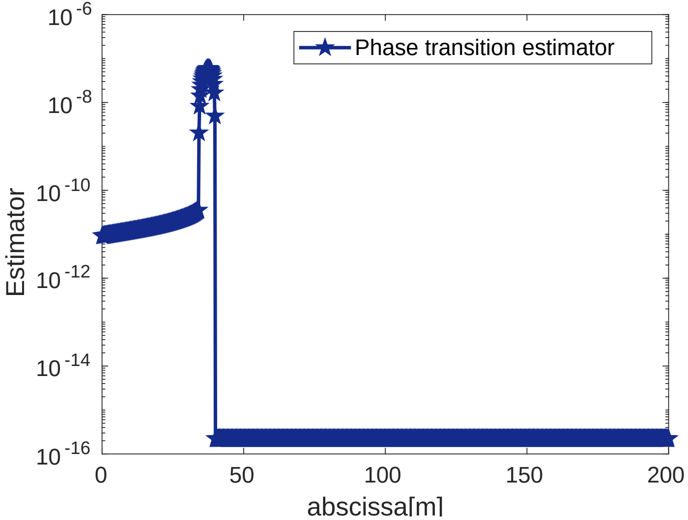
t = 2500 years

Phase transition estimator



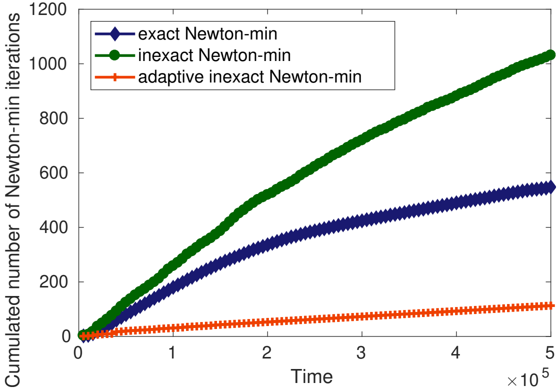
$t = 1.25 \times 10^4$ **years**

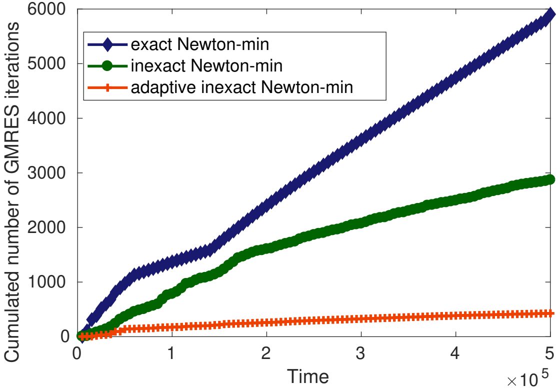
Phase transition estimator



$t = 4.25 \times 10^4$ years

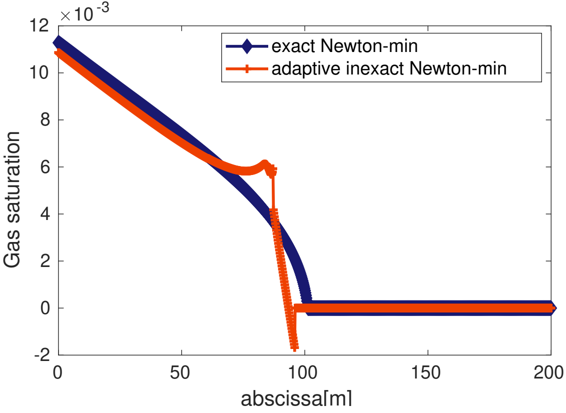
Overall performance $\gamma_{\mathrm{lin}} = \gamma_{\mathrm{alg}} = 10^{-3}$



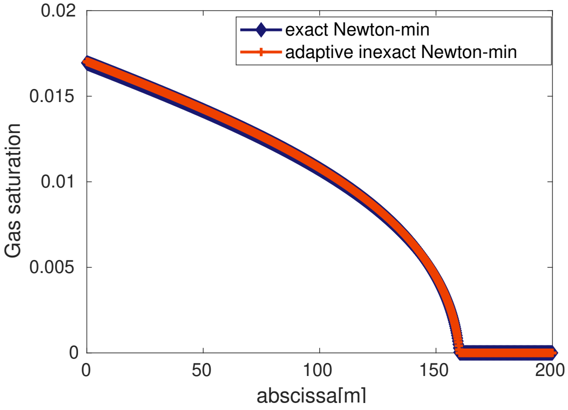


Accuracy $\gamma_{\text{lin}} = \gamma_{\text{alg}} = 10^{-3}$

$t = 1.05 \times 10^5$ years



$t = 3.5 \times 10^5$ years



Outline

Conclusion

- Variational inequality: we devised a posteriori error estimates with $\mathbb{P}_p$ finite elements.
- Two-phase flow with phase transition: a posteriori error estimates for a cell centered finite volume discretization.
- Formulations with complementarity constraints and semismooth algorithms.
- We distinguished the different error components.
- Adaptive stopping criteria $\Rightarrow$ reduction of the number of iterations.
- Extension of the stationary problem to a parabolic inequality:
  J. Dabaghi, V. Martin, and M. Vohralík, *A posteriori error estimate and adaptive stopping criteria for a parabolic variational inequality*. In preparation, 2019

Implementation

- contact problem between two membranes: MATLAB code. Collaboration with Jan Papež (INRIA Paris).
- Two phase flow: MATLAB code. Collaboration with Ibtihel Ben Gharbia (IFPEN).

Thank you for your attention