Adaptive inexact semismooth Newton methods for the contact problem between two membranes

Jad Dabaghi, Vincent Martin, Martin Vohralík

Inria Paris & Université Paris-Est

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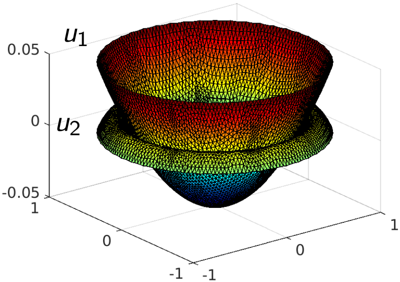
Outline

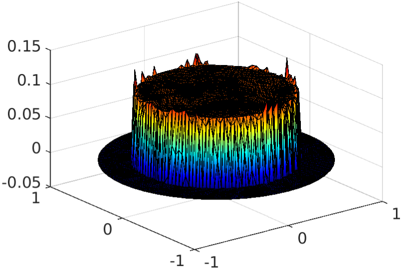
Introduction

System of variational inequalities:

Find u_1, u_2, λ such that

$$\begin{cases} -\mu_1 \Delta u_1 - \lambda = f_1 & \text{in} \quad \Omega, \\ -\mu_2 \Delta u_2 + \lambda = f_2 & \text{in} \quad \Omega, \\ (u_1 - u_2)\lambda = 0, \quad u_1 - u_2 \geq 0, \quad \lambda \geq 0 & \text{in} \quad \Omega, \\ u_1 = g > 0 & \text{on} \quad \partial\Omega, \\ u_2 = 0 & \text{on} \quad \partial\Omega. \end{cases}$$





Continuous model problem and setting

- $H^1_g(\Omega)=\{u \in H^1(\Omega),\ u = g \text{ on } \partial\Omega\}$ $\quad \Lambda = \{\chi \in L^2(\Omega),\ \chi \geq 0 \text{ on } \Omega\}$
- $\mathcal{K}^g = \{(v_1, v_2) \in H^1_g(\Omega) \times H^1_0(\Omega),\ v_1 - v_2 \geq 0 \text{ on } \Omega\}$

Weak formulation: For $(f_1, f_2) \in L^2(\Omega) \times L^2(\Omega)$ and $g > 0$ find $(u_1, u_2, \lambda) \in H^1_g(\Omega) \times H^1_0(\Omega) \times \Lambda$ such that

$$\left\{ \begin{array}{l} \sum_{\alpha=1}^{2} \mu_i \left(\nabla u_i, \nabla v_i\right)_\Omega - (\lambda, v_1 - v_2)_\Omega = \sum_{\alpha=1}^{2} (f_i, v_i)_\Omega \quad \forall (v_1, v_2) \in \left(H^1_0(\Omega)\right)^2 \\ (\chi - \lambda, u_1 - u_2)_\Omega \geq 0 \quad \forall \chi \in \Lambda. \end{array} \right.$$

equivalent to

Reduced problem:

$$\sum_{\alpha=1}^{2} \mu_i \left(\nabla u_i, \nabla (v_i - u_i)\right)_\Omega \geq \sum_{\alpha=1}^{2} (f_i, v_i - u_i)_\Omega \quad \forall \boldsymbol{v} = (v_1, v_2) \in \mathcal{K}^g.$$

Existence and uniqueness based on Lions-Stamppachia Theorem (Ben Belgacem *et al.* 2008)

Discretization by finite elements

Notation: $\mathcal{T}_h$: conforming mesh, $\mathcal{V}_{\mathrm{d}}^p$: set of DOFs, $\mathcal{N}_{\mathrm{d}}^p$ number of DOFs, $\mathcal{V}_h$: set of vertices

Spaces for the discretization:

$$X_{gh}^p = \{v_h \in \mathcal{C}^0(\overline{\Omega}), v_{h|K} \in \mathbb{P}_p(K), \ \forall K \in \mathcal{T}_h, \ \ v_h = g \ \text{ on } \ \partial\Omega\}$$

$$X_{0h}^p = \{v_h \in \mathcal{C}^0(\overline{\Omega}); \ v_h|_K \in \mathbb{P}_p(K), \quad \forall K \in \mathcal{T}_h\} \cap H_0^1(\Omega)$$

$$\mathcal{K}_{gh}^p = \left\{(v_{1h}, v_{2h}) \in X_{gh}^p \times X_{0h}^p, \ v_{1h}(\boldsymbol{x}_l) - v_{2h}(\boldsymbol{x}_l) \geq 0 \ \ \forall \boldsymbol{x}_l \in \mathcal{V}_{\mathrm{d}}^p\right\} \not\subset \mathcal{K}^g$$

Discrete reduced problem: find $\boldsymbol{u}_h = (u_{1h}, u_{2h}) \in \mathcal{K}_{gh}^p$ such that

$$\sum_{\alpha=1}^{2} \mu_i \left(\boldsymbol{\nabla} u_{ih}, \boldsymbol{\nabla}\left(v_{ih} - u_{ih}\right)\right)_\Omega \geq \sum_{\alpha=1}^{2} (f_i, v_{ih} - u_{ih})_\Omega \quad \forall \boldsymbol{v}_h = (v_{1h}, v_{2h}) \in \mathcal{K}^g.$$

Resolution techniques: Projected Newton methods (Bertsekas 1982), Active set Newton method (Kanzow 1999), Primal-dual active set strategy (Hintermüller 2002).

Characterization of the discrete lagrange multiplier:

$$\begin{cases} \langle \lambda_{1h}, v_{1h} \rangle_h = \mu_1 (\nabla u_{1h}, \nabla v_{1h})_\Omega - (f_1, v_{1h})_\Omega & \forall v_{1h} \in X_{0h}^p, \\ \langle \lambda_{2h}, v_{2h} \rangle_h = -\mu_2 (\nabla u_{2h}, \nabla v_{2h})_\Omega + (f_2, v_{2h})_\Omega & \forall v_{2h} \in X_{0h}^p, \end{cases}$$

where $\forall (w_h, v_h) \in X_{0h}^p \times X_{0h}^p$

$$\langle w_h, v_h \rangle_h = \begin{cases} \sum_{\boldsymbol{a} \in \mathcal{V}_h^{\mathrm{int}}} w_h(\boldsymbol{a}) v_h(\boldsymbol{a}) (\psi_{h,\boldsymbol{a}}, 1)_\Omega & \text{if} \quad \text{p=1}, \\ (w_h, v_h)_\Omega & \text{if} \quad \text{p} \geq 2 \end{cases} \tag{1}$$

Lemma

- *The functions λ_{1h} and λ_{2h} coincide. We set $\lambda_h = \lambda_{1h} = \lambda_{2h}$.*
- $\langle \lambda_h, \psi_{h,\mathbf{x}_l} \rangle_h \geq 0.$

Definition

$$\Lambda_h^p = \left\{ v_h \in X_{0h}^p ; \ \langle v_h, \psi_{h,\mathbf{x}_l} \rangle_h \geq 0 \quad \forall (\psi_{h,\mathbf{x}_l})_{1 \leq l \leq \mathcal{N}_{\mathrm{d}}^p} \in X_{0h}^p \right\}.$$

Application to $\mathbb{P}_1$ finite elements

Conforming spaces

- $\mathcal{K}^1_{gh} = \{(v_{1h}, v_{2h}) \in X^1_{gh} \times X^1_{0h},\ v_{1h}(\boldsymbol{a}) - v_{2h}(\boldsymbol{a}) \geq 0 \quad \forall \boldsymbol{a} \in \mathcal{V}_h\} \subset \mathcal{K}^g$
- $\Lambda^1_h = \{v_h \in X^1_{0h};\ v_h(\boldsymbol{a}) \geq 0 \quad \forall \boldsymbol{a} \in \mathcal{V}_h^{\mathrm{int}}\} \subset \Lambda$

Weak formulation (Ben Belgacem *et al.* 2008)

find $(u_{1h}, u_{2h}, \lambda_h) \in X^1_{gh} \times X^1_{0h} \times \Lambda^1_h$ s.t $\forall (v_{1h}, v_{2h}, \chi_h) \in X^1_{0h} \times X^1_{0h} \times \Lambda^1_h$

$$\sum_{\alpha=1}^{2} \mu_\alpha (\nabla u_{\alpha h}, \nabla v_{\alpha h})_\Omega - \sum_{\boldsymbol{a} \in \mathcal{V}_h^{\mathrm{int}}} \lambda_h(\boldsymbol{a})(v_{1h} - v_{2h})(\boldsymbol{a})\, (\psi_{h,\boldsymbol{a}}, 1)_\Omega = \sum_{\alpha=1}^{2} (f_\alpha, v_{\alpha h})_\Omega$$

$$(u_{1h} - u_{2h})(\boldsymbol{a}) \geq 0,\ \lambda_h(\boldsymbol{a}) \geq 0,\ \lambda_h(\boldsymbol{a})(u_{1h} - u_{2h})(\boldsymbol{a}) = 0.$$

Can we reformulate the discrete constraints?

Discrete complementarity problem

Definition

A function $f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is a C-function if

$$\forall(\boldsymbol{a}, \boldsymbol{b}) \in \mathbb{R}^n \times \mathbb{R}^n \quad f(\boldsymbol{a}, \boldsymbol{b}) = 0 \quad \Longleftrightarrow \quad \boldsymbol{a} \geq 0, \quad \boldsymbol{b} \geq 0, \quad \boldsymbol{ab} = 0.$$

For any C-function $\mathbf{C}$, the discretization reads

$$\begin{cases} \mathbb{E}\boldsymbol{X}_h & = \boldsymbol{F} \\ \mathbf{C}(\boldsymbol{X}_h) & = 0. \end{cases}$$

C is not Fréchet differentiable!

Example: semismooth "min" function

$$\mathbf{C}(\boldsymbol{X}_{1h} - \boldsymbol{X}_{2h}, \boldsymbol{X}_{3h}) = \min\left(\boldsymbol{X}_{1h} - \boldsymbol{X}_{2h}, \boldsymbol{X}_{3h}\right)$$

Example: semismooth "Fischer-Burmeister" function

$$\mathbf{C}(\boldsymbol{X}_{1h} - \boldsymbol{X}_{2h}, \boldsymbol{X}_{3h}) = \sqrt{(\boldsymbol{X}_{1h} - \boldsymbol{X}_{2h})^2 + \boldsymbol{X}_{3h}^2} - (\boldsymbol{X}_{1h} - \boldsymbol{X}_{2h} + \boldsymbol{X}_{3h})$$

The vector of unknowns has the following block structure

$$\boldsymbol{X}_h^T = (\boldsymbol{X}_{1h}, \boldsymbol{X}_{2h}, \boldsymbol{X}_{3h})^T \in \mathbb{R}^{3N_h}$$

Semismooth Newton method

For $\boldsymbol{X}_h^0$ given, the semismooth Newton method reads

$$\mathbb{A}^{k-1}\boldsymbol{X}_h^k = \boldsymbol{B}^{k-1} \quad \forall k \geq 1$$

The Jacobian matrix (element of the Clarke subdifferential) and the right-hand side vector are defined by

$$\mathbb{A}^{k-1} = \begin{cases} \mathbb{E} \\ \mathbf{J_C}(\boldsymbol{X}_h^{k-1}) \end{cases} \text{and } \boldsymbol{B}^{k-1} = \begin{cases} \boldsymbol{F} \\ \mathbf{J_C}(\boldsymbol{X}_h^{k-1})\boldsymbol{X}_h^{k-1} - \mathbf{C}(\boldsymbol{X}_h^{k-1}) \end{cases} \forall k \geq 1.$$

Example: Jacobian matrix for the "min" function

$$\underbrace{\begin{pmatrix} 1 & \cdots & 0 & -1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 1 & 0 & \cdots & -1 & 0 & \cdots & 0 \end{pmatrix}}_{=\mathbb{K}} \quad \underbrace{\begin{pmatrix} 0 & \cdots & 0 & 0 & \cdots & 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 1 \end{pmatrix}}_{=\mathbb{G}}$$

$$\mathbf{J_C}(\boldsymbol{X}_h^k)_l = \begin{cases} \mathbb{K}_l & \text{if} \quad u_{1h}^k(\boldsymbol{a}_l) - u_{2h}^k(\boldsymbol{a}_l) \leq \lambda_h^k(\boldsymbol{a}_l) \\ \mathbb{G}_l & \text{if} \quad \lambda_h^k(\boldsymbol{a}_l) < u_{1h}^k(\boldsymbol{a}_l) - u_{2h}^k(\boldsymbol{a}_l) \end{cases}$$

Algebraic resolution in semismooth Newton method

Any iterative algebraic solver yields on step $i \geq 0$:

$$\mathbb{A}^{k-1}\mathbf{X}_h^{k,i} + \boldsymbol{R}_h^{k,i} = \boldsymbol{B}^{k-1}$$

with $\boldsymbol{R}_h^{k,i} = (\boldsymbol{R}_{1h}^{k,i}, \boldsymbol{R}_{2h}^{k,i}, \boldsymbol{R}_{3h}^{k,i})^T$ the algebraic residual block vector.

Definition

We define discontinous $\mathbb{P}_1$ polynomials $r_{1h}^{k,i}$ and $r_{2h}^{k,i}$

- $(r_{1h}^{k,i}, \psi_{h,\boldsymbol{a}_l})_K = \dfrac{(\boldsymbol{R}_{1h}^{k,i})_l}{N_{h,\boldsymbol{a}}}, \quad r_{1h}^{k,i}{}_{|\partial K \cap \partial\Omega} = 0 \qquad \forall 1 \leq l \leq N_h$
- $(r_{2h}^{k,i}, \psi_{h,\boldsymbol{a}_l})_K = \dfrac{(\boldsymbol{R}_{2h}^{k,i})_l}{N_{h,\boldsymbol{a}}}, \quad r_{2h}^{k,i}{}_{|\partial K \cap \partial\Omega} = 0 \qquad \forall 1 \leq l \leq N_h$

Equivalent form of the $2N_h$ first equations

$$\mu_1 \left(\boldsymbol{\nabla} u_{1h}^{k,i}, \boldsymbol{\nabla}\psi_{h,\boldsymbol{a}_l}\right)_\Omega = \left(f_1 + \lambda_h^{k,i}(\boldsymbol{a}_l) - r_{1h}^{k,i}, \psi_{h,\boldsymbol{a}_l}\right)_\Omega,$$
$$\mu_2 \left(\boldsymbol{\nabla} u_{2h}^{k,i}, \boldsymbol{\nabla}\psi_{h,\boldsymbol{a}_l}\right)_\Omega = \left(f_2 - \lambda_h^{k,i}(\boldsymbol{a}_l) - r_{2h}^{k,i}, \psi_{h,\boldsymbol{a}_l}\right)_\Omega.$$

A posteriori analysis and preliminary study

A posteriori error estimates: $|||\boldsymbol{u}-\boldsymbol{u}_h^{k,i}||| \leq \left\{ \sum_{K\in\mathcal{T}_h} \eta_K(\boldsymbol{u}_h^{k,i})^2 \right\}^{1/2}.$

General introduction: Ainsworth & Oden (2000), Repin (2008), Verfürth (2013). Obsacle problems: Veeser (2001), Chen & Nochetto (2000), Bartels & Carstensen (2004).

Goal:
$$\begin{cases} \boldsymbol{\sigma}_{1h}^{k,i} \in \mathbf{H}(\mathrm{div},\Omega) \text{ such that } (\boldsymbol{\nabla}\cdot\boldsymbol{\sigma}_{1h}^{k,i},1)_K = (f_1+\lambda_h^{k,i},1)_K \ \forall K\in\mathcal{T}_h, \\ \boldsymbol{\sigma}_{2h}^{k,i} \in \mathbf{H}(\mathrm{div},\Omega) \text{ such that } (\boldsymbol{\nabla}\cdot\boldsymbol{\sigma}_{2h}^{k,i},1)_K = (f_2-\lambda_h^{k,i},1)_K \ \forall K\in\mathcal{T}_h. \end{cases}$$

- $\boldsymbol{\sigma}_{1h}^{k,i} = \boldsymbol{\sigma}_{1h,\mathrm{disc}}^{k,i} + \boldsymbol{\sigma}_{1h,\mathrm{alg}}^{k,i}$ and $\boldsymbol{\sigma}_{2h}^{k,i} = \boldsymbol{\sigma}_{2h,\mathrm{disc}}^{k,i} + \boldsymbol{\sigma}_{2h,\mathrm{alg}}^{k,i}$

Algebraic fluxes reconstruction:

- $\left\{\boldsymbol{\sigma}_{1h,\mathrm{alg}}^{k,i}, \boldsymbol{\sigma}_{2h,\mathrm{alg}}^{k,i}\right\} \in \mathbf{H}(\mathrm{div},\Omega), \quad \boldsymbol{\nabla}\cdot\boldsymbol{\sigma}_{1h,\mathrm{alg}}^{k,i} = r_{1h}^{k,i}, \quad \boldsymbol{\nabla}\cdot\boldsymbol{\sigma}_{2h,\mathrm{alg}}^{k,i} = r_{2h}^{k,i}$

Papez Jan, Rüde Ulrich, Vohralík Martin, and Wohlmuth Barbara.
Sharp algebraic and total a posteriori error bounds via a multilevel approach.
Submitted, 2017.

Discretization fluxes reconstruction

$\boldsymbol{\sigma}_{1h,\mathrm{disc}}^{k,i,\boldsymbol{a}}$ and $\boldsymbol{\sigma}_{2h,\mathrm{disc}}^{k,i,\boldsymbol{a}}$ are the solution of mixed system on patches

$$\begin{cases} (\boldsymbol{\sigma}_{1h,\mathrm{disc}}^{k,i,\boldsymbol{a}}, \boldsymbol{v}_{1h})_{\omega_h^{\boldsymbol{a}}} - (\gamma_{1h}^{k,i,\boldsymbol{a}}, \nabla\cdot\boldsymbol{v}_{1h})_{\omega_h^{\boldsymbol{a}}} & = -\left(\psi_{h,\boldsymbol{a}}\nabla u_{1h}^{k,i}, \boldsymbol{v}_{1h}\right)_{\omega_h^{\boldsymbol{a}}} \quad \forall \boldsymbol{v}_{1h}\in \mathbf{V}_h^{\boldsymbol{a}},\\ (\nabla\cdot\boldsymbol{\sigma}_{1h,\mathrm{disc}}^{k,i,\boldsymbol{a}}, q_{1h})_{\omega_h^{\boldsymbol{a}}} & = (\tilde{g}_{1h}^{k,i,\boldsymbol{a}}, q_{1h})_{\omega_h^{\boldsymbol{a}}} \quad \forall q_{1h}\in Q_h^{\boldsymbol{a}},\\ (\boldsymbol{\sigma}_{2h,\mathrm{disc}}^{k,i,\boldsymbol{a}}, \boldsymbol{v}_{2h})_{\omega_h^{\boldsymbol{a}}} - (\gamma_{2h}^{k,i,\boldsymbol{a}}, \nabla\cdot\boldsymbol{v}_{2h})_{\omega_h^{\boldsymbol{a}}} & = -\left(\psi_{h,\boldsymbol{a}}\nabla u_{2h}^{k,i}, \boldsymbol{v}_{2h}\right)_{\omega_h^{\boldsymbol{a}}} \quad \forall \boldsymbol{v}_{2h}\in \mathbf{V}_h^{\boldsymbol{a}},\\ (\nabla\cdot\boldsymbol{\sigma}_{2h,\mathrm{disc}}^{k,i,\boldsymbol{a}}, q_{2h})_{\omega_h^{\boldsymbol{a}}} & = (\tilde{g}_{2h}^{k,i,\boldsymbol{a}}, q_{2h})_{\omega_h^{\boldsymbol{a}}} \quad \forall q_{2h}\in Q_h^{\boldsymbol{a}}. \end{cases}$$

$$\boldsymbol{\sigma}_{1h,\mathrm{disc}}^{k,i} = \sum_{\boldsymbol{a}\in\mathcal{V}_h} \boldsymbol{\sigma}_{1h,\mathrm{disc}}^{k,i,\boldsymbol{a}} \quad \text{and} \quad \boldsymbol{\sigma}_{2h,\mathrm{disc}}^{k,i} = \sum_{\boldsymbol{a}\in\mathcal{V}_h} \boldsymbol{\sigma}_{2h,\mathrm{disc}}^{k,i,\boldsymbol{a}}.$$

- $\boldsymbol{\sigma}_{1h,\mathrm{disc}}^{k,i} \in \mathbf{H}(\mathrm{div},\Omega)$ and $\left(\nabla\cdot\boldsymbol{\sigma}_{1h,\mathrm{disc}}^{k,i}, 1\right)_K = \left(f_1 + \lambda_h^{k,i} - r_{1h}^{k,i}, 1\right)_K$
- $\boldsymbol{\sigma}_{2h,\mathrm{disc}}^{k,i} \in \mathbf{H}(\mathrm{div},\Omega)$ and $\left(\nabla\cdot\boldsymbol{\sigma}_{2h,\mathrm{disc}}^{k,i}, 1\right)_K = \left(f_2 - \lambda_h^{k,i} - r_{2h}^{k,i}, 1\right)_K$.

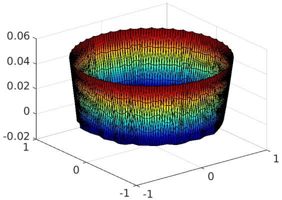
Destuynder & Métivet (1999), Braess & Schöberl (2009), Ern & Vohralík (2013).

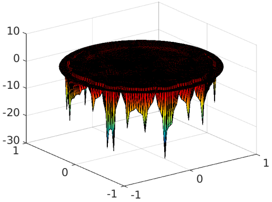
A posteriori error estimates

$\boldsymbol{u} = (u_1, u_2) \in \mathcal{K}^g$, $\boldsymbol{u}_h^{k,i} = (u_{1h}^{k,i}, u_{2h}^{k,i}) \in X_{gh}^p \times X_{0h}^p$, $\left(\boldsymbol{\sigma}_{1h}^{k,i}, \boldsymbol{\sigma}_{2h}^{k,i}\right) \in \mathbf{H}(\mathrm{div}, \Omega)$

Warning: $u_{1h}^{k,i}(\boldsymbol{x}_l) - u_{2h}^{k,i}(\boldsymbol{x}_l)$ and $\left\langle \lambda_h^{k,i}, \psi_{h,\boldsymbol{x}_l} \right\rangle_h$ can be negative.

Example: $\mathbb{P}_1$ **discretization** $\Rightarrow$ $u_{1h}^{k,i}(\boldsymbol{a}) - u_{2h}^{k,i}(\boldsymbol{a}) \leq 0$ and $\lambda_h^{k,i}(\boldsymbol{a}) \leq 0$





Motivation: $\widetilde{\mathcal{K}}_{gh}^p = \left\{(v_{1h}, v_{2h}) \in X_{gh}^p \times X_{0h}^p,\ v_{1h} - v_{2h} \geq 0\right\} \subset \mathcal{K}^g$.

Define $\boldsymbol{s}_h^{k,i} \in \widetilde{\mathcal{K}}_{gh}^1 = \mathcal{K}_{gh}^1$ by $\boldsymbol{s}_h^{k,i}(\boldsymbol{a}) =$

$$\begin{cases} \boldsymbol{u}_h^{k,i}(\boldsymbol{a}) = \left(u_{1h}^{k,i}(\boldsymbol{a}), u_{2h}^{k,i}(\boldsymbol{a})\right) & \text{if } u_{1h}^{k,i}(\boldsymbol{a}) \geq u_{2h}^{k,i}(\boldsymbol{a}), \\ \left(1/2(u_{1h}^{k,i}(\boldsymbol{a}) + u_{2h}^{k,i}(\boldsymbol{a})), 1/2(u_{1h}^{k,i}(\boldsymbol{a}) + u_{2h}^{k,i}(\boldsymbol{a}))\right) & \text{if } u_{1h}^{k,i}(\boldsymbol{a}) < u_{2h}^{k,i}(\boldsymbol{a}). \end{cases}$$

Discretization error estimators

$$\left.\begin{aligned}
\eta_{\mathrm{F},K,\alpha}^{k,i} &= \left\| \mu_\alpha^{\frac{1}{2}} \nabla u_{\alpha h}^{k,i} + \mu_\alpha^{-\frac{1}{2}} \boldsymbol{\sigma}_{\alpha h,\mathrm{disc}}^{k,i} \right\|_K \\
\eta_{\mathrm{osc},K,\alpha} &= \frac{h_K}{\pi} \mu_\alpha^{-\frac{1}{2}} \left\| f_\alpha - \boldsymbol{\Pi}_{\mathbb{P}_1}(f_\alpha) \right\|_K \\
\eta_{\mathrm{C},K}^{k,i,\mathrm{pos}} &= 2 \left(u_{1h}^{k,i} - u_{2h}^{k,i}, \lambda_h^{k,i,\mathrm{pos}} \right)_K
\end{aligned}\right\} \Rightarrow \boldsymbol{\eta}_{\mathrm{disc}}^{k,i}$$

Linearization error estimators

$$\left.\begin{aligned}
\eta_{\mathrm{lin},1,K}^{k,i} &= \left|\left|\left| \boldsymbol{s}_h^{k,i} - \boldsymbol{u}_h^{k,i} \right|\right|\right|_K \\
\eta_{\mathrm{lin},2,K}^{k,i} &= 2 h_\Omega C_{\mathrm{PF}} \left(\frac{1}{\mu_1} + \frac{1}{\mu_2} \right)^{\frac{1}{2}} \left\| \lambda_h^{k,i,\mathrm{pos}} \right\|_\Omega \left|\left|\left| \boldsymbol{s}_h^{k,i} - \boldsymbol{u}_h^{k,i} \right|\right|\right|_K \\
\eta_{\mathrm{lin},3,K}^{k,i} &= h_\Omega C_{\mathrm{PF}} \left(\frac{1}{\mu_1} + \frac{1}{\mu_2} \right)^{\frac{1}{2}} \left\| \lambda_h^{k,i,\mathrm{neg}} \right\|_K
\end{aligned}\right\} \Rightarrow \boldsymbol{\eta}_{\mathrm{lin}}^{k,i}$$

Algebraic error estimators

$$\left.\eta_{\mathrm{alg},K,\alpha}^{k,i} = \left\| \mu_\alpha^{-\frac{1}{2}} \boldsymbol{\sigma}_{\alpha h,\mathrm{alg}}^{k,i} \right\|_K \right\} \Rightarrow \boldsymbol{\eta}_{\mathrm{alg}}^{k,i}$$

Theorem (A posteriori estimate distinguishing the error components)

$$\left|\left|\left| \boldsymbol{u} - \boldsymbol{u}_h^{k,i} \right|\right|\right| \le \boldsymbol{\eta}_{\mathrm{disc}}^{k,i} + \boldsymbol{\eta}_{\mathrm{alg}}^{k,i} + \boldsymbol{\eta}_{\mathrm{lin}}^{k,i}.$$

Adaptive inexacte semismooth Newton algorithm

Algorithm 1 Adaptive inexact semismooth Newton algorithm

Initialization: Choose an initial vector $\boldsymbol{X}_h^0 \in \mathcal{M}_{3N_h,1}(\mathbb{R})$, $(k = 0)$

Do

$k = k + 1$

Compute $\mathbb{A}^{k-1} \in \mathcal{M}_{3N_h,3N_h}(\mathbb{R})$, $\boldsymbol{B}^{k-1} \in \mathcal{M}_{3N_h,1}(\mathbb{R})$

Consider $\mathbb{A}^{k-1}\mathbf{X}_h^k = \boldsymbol{B}^{k-1}$

Initialization for the linear solver: Define $\boldsymbol{X}_h^{k,0} = \boldsymbol{X}_h^{k-1}$, $(i = 0)$

Do

$i = i + 1$

Compute Residual: $\boldsymbol{R}_h^{k,i} = \boldsymbol{B}^{k-1} - \mathbb{A}^{k-1}\mathbf{X}_h^{k,i}$

Compute estimators

While $\boldsymbol{\eta}_{\mathrm{alg}}^{k,i} \geq \gamma_{\mathrm{alg}} \max\left\{\eta_{\mathrm{disc}}^{k,i}, \eta_{\mathrm{lin}}^{k,i}\right\}$

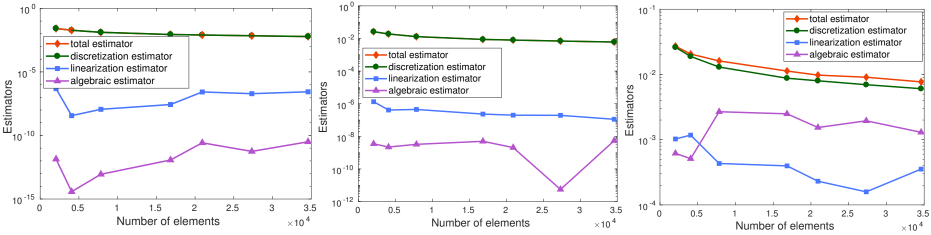
While $\boldsymbol{\eta}_{\mathrm{lin}}^{k,i} \geq \gamma_{\mathrm{lin}}\eta_{\mathrm{disc}}^{k,i}$

End

Numerical experiments

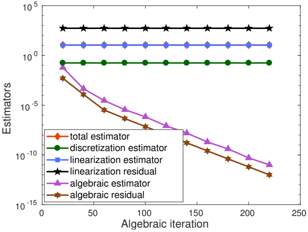
- Ω = unit disk, $J = 3$, $\mu_1 = \mu_2 = 1$, $g = 0.05$, $\gamma_{\text{lin}} = 0.3$ $\gamma_{\text{alg}} = 0.3$
- semismooth solver: Newton-min. Linear solver: GMRES

Exact Newton | Inexact Newton | Adaptive inexact Newton

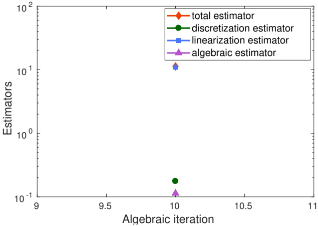


Quality and precision are preserved for adaptive inexact semismooth Newton method.

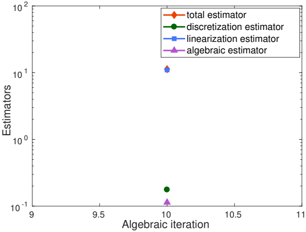
Exact Newton



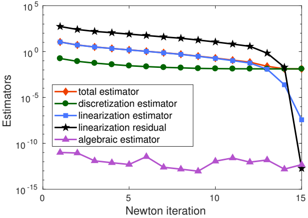
Inexact Newton



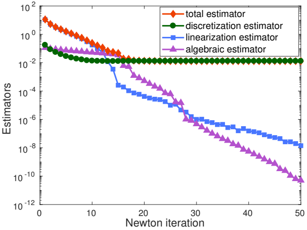
Adaptive inexact Newton



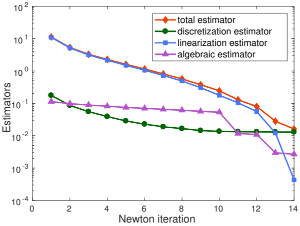
Exact Newton



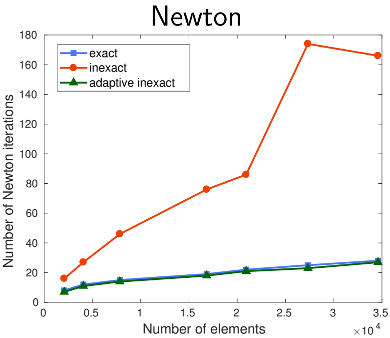
Inexact Newton

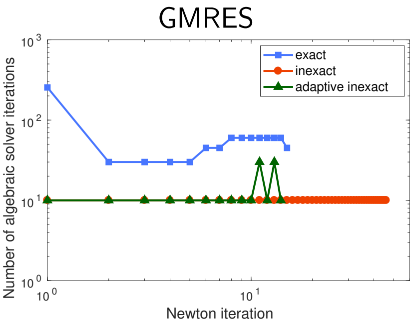


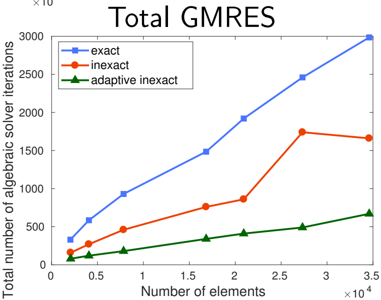
Adaptive Inexact Newton

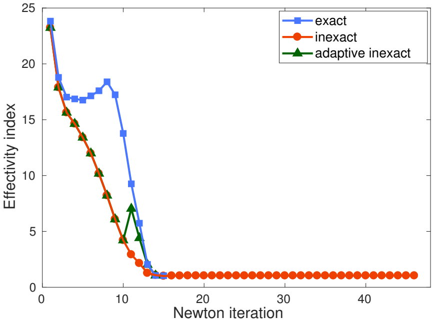


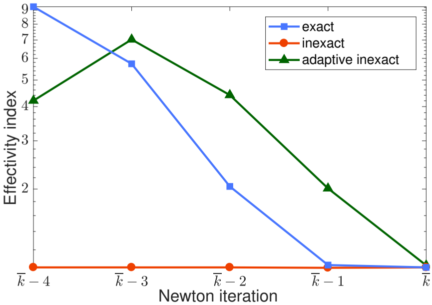
Overall performance of the three approaches:

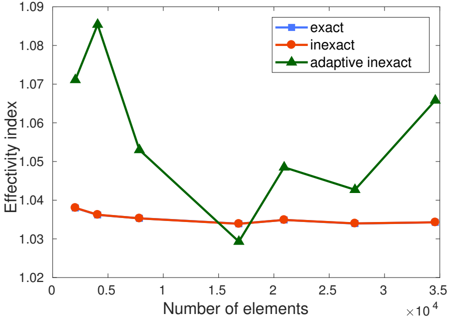




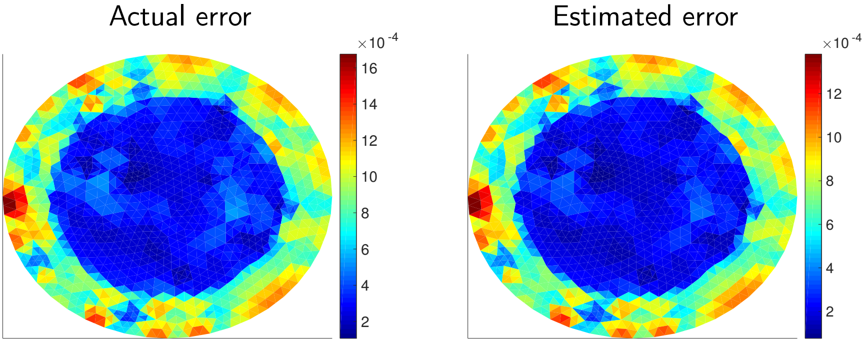



exact
inexact
adaptive inexact
Effectivity index
Newton iteration
0
5
10
15
20
25
30
40


exact
inexact
adaptive inexact
Effectivity index
Newton iteration
$\overline{k}-4$
$\overline{k}-3$
$\overline{k}-2$
$\overline{k}-1$
$\overline{k}$


exact
inexact
adaptive inexact
Effectivity index
Number of elements
$\times 10^4$

Distribution of the error:



Conclusion

- We devised an a posteriori error estimate between $\boldsymbol{u}$ and $\boldsymbol{u}_h^{k,i}$ for a wide class of semismooth Newton methods.
- This estimate enables to control the error at each semismooth Newton step.
- The adaptive inexact semismooth Newton method requires less nonlinear and linear steps.
- Ongoing work: extension to unsteady problems with nonlinear complementarity constraints

J. Dabaghi, V. Martin, and M. Vohralík, *Adaptive inexact semismooth Newton methods for the contact problem between two membranes*. submitted for publication.

I. Ben Gharbia, J. Dabaghi, V. Martin, and M. Vohralík, *A posteriori error estimates and adaptive stopping criteria, for a compositional two-phase flow with nonlinear complementarity constraints*. In preparation, 2018.